No Eigenvalues Outside the Support of the Limiting Spectral Distribution of Large Dimensional Sample Covariance Matrices

By Z.D. Bai[1] and Jack W. Silverstein[2]

National University of Singapore and North Carolina State University

Dedicated to the Sixtieth Birthday of Min-Te Chao

Summary

Let $B_n = (1/N)T_n^{1/2}X_nX_n^*T_n^{1/2}$ where X_n is $n \times N$ with i.i.d. complex standardized entries having finite fourth moment, and $T_n^{1/2}$ is a Hermitian square root of the nonnegative definite Hermitian matrix T_n. It is known that, as $n \to \infty$, if n/N converges to a positive number, and the empirical distribution of the eigenvalues of T_n converges to a proper probability distribution, then the empirical distribution of the eigenvalues of B_n converges $a.s.$ to a nonrandom limit. In this paper we prove that, under certain conditions on the eigenvalues of T_n, for any closed interval outside the support of the limit, with probability 1 there will be no eigenvalues in this interval for all n sufficiently large.

1. Introduction

For $n = 1, 2, \ldots$ let $X = X_n = (X_{ij})$, $T = T_n$, and $T_n^{1/2}$, denote, respectively, an $n \times N$ matrix consisting of i.i.d. standardized complex entries ($\mathsf{E}X_{11} = 0$, $\mathsf{E}|X_{11}|^2 = 1$), an $n \times n$ nonnegative definite matrix, and any square root of T. For any square matrix A having real eigenvalues, let F^A denote the empirical distribution function (e.d.f.) of its eigenvalues. The matrix $B_n = (1/N)T_n^{1/2}XX^*T_n^{1/2}$ can be viewed as the sample covariance matrix of a broad class of random vectors, $T_n^{1/2}X_{\cdot 1}$ ($X_{\cdot j}$ denoting the jth column of X). Previous work on understanding the behavior of the eigenvalues of B_n when n and N are large but have the same order of magnitude has been on F^{B_n} and on the extreme eigenvalues when $T = I$, the identity matrix. Assuming $N = N(n)$ with $n/N \to c > 0$ as $n \to \infty$ and $F^T \overset{\mathcal{D}}{\longrightarrow} H$, a proper p.d.f. it is known that almost surely F^{B_n} converges weakly to a nonrandom p.d.f. F (see Silverstein (1995)). Proving this result, along with describing F (which can be explicitly expressed in only a few cases), is best achieved with the aid of the Stieltjes transform, defined for any p.d.f. G by

$$m_G(z) \equiv \int \frac{1}{\lambda - z} dG(\lambda) \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : Im\, z > 0\}.$$

Because of the inversion formula

$$G([a,b]) = \frac{1}{\pi} \lim_{\eta \to 0} \int_a^b Im\, m_G(\xi + i\eta) d\xi$$

(a, b continuity points of G), weak convergence of p.d.f.'s can be proven by showing convergence of Stieltjes transforms.

For each $z \in \mathbb{C}^+$, $m = m_F(z)$ is a solution to the equation

$$m = \int \frac{1}{t(1 - c - czm) - z} dH(t),$$

which is unique in the set $\{m \in \mathbb{C} : -(1-c)/z + cm \in \mathbb{C}^+\}$. Let $\underline{B}_n = (1/N)X^*TX$. Since the spectra of B_n and $\underline{B}_n$ differ by $|n - N|$ zero eigenvalues, it follows that

$$F^{\underline{B}_n} = \left(1 - \frac{n}{N}\right)I_{[0,\infty)} + \frac{n}{N}F^{B_n},$$

from which we get

$$m_{F^{\underline{B}_n}}(z) = -\frac{(1 - n/N)}{z} + \frac{n}{N}m_{F^{B_n}}(z) \quad z \in \mathbb{C}^+,$$

and with $\underline{F}$ denoting the limit of $F^{\underline{B}_n}$ we have

$$\underline{F} = (1 - c)I_{[0,\infty)} + cF,$$

1

and

$$m_{\underline{F}}(z) = -\frac{(1-c)}{z} + cm_F(z) \quad z \in \mathbb{C}^+.$$

It follows that

$$m_F = -z^{-1} \int \frac{1}{1 + tm_{\underline{F}}} dH(t),$$

for each $z \in \mathbb{C}^+$, $\underline{m} = m_{\underline{F}}(z)$, is the unique solution in $\mathbb{C}^+$ to the equation

$$\underline{m} = -\left(z - c \int \frac{t\,dH(t)}{1 + t\underline{m}}\right)^{-1}, \tag{1.1}$$

and $m_{\underline{F}}(z)$ has an inverse, explicitly given by

$$z(\underline{m}) = -\frac{1}{\underline{m}} + c \int \frac{t\,dH(t)}{1 + t\underline{m}}. \tag{1.2}$$

Much of the analytic behavior of F can be inferred from these equations (see Silverstein and Choi (1995)). Indeed, continuous dependence of F on c and H is readily apparent from (1.2), and the inversion formula, and it can be shown that $F \xrightarrow{\mathcal{D}} H$ as $c \to 0$. Moreover, it is shown in Silverstein and Choi (1995) that, away from zero, F has a continuous density. As an example Figure 1 (a) is the graph of the density when $c = .1$ and H places mass .2, .4, and .4 at, respectively, 1, 3, and 10.

The focus of this paper is on intervals of $\mathbb{R}^+$ lying outside the support of F. The inverse (1.2) can be used to identify these intervals, mainly because, on any such interval, $m_{\underline{F}}$ exists and is increasing. Consequently its inverse will also exist and will be increasing on the range of this interval. Silverstein and Choi (1995) confirms each $\underline{m}$ in this range is such that $-1/\underline{m}$ lies outside the support of H. Therefore, plotting (1.2) on $\mathbb{R}$ and observing the range of values where it is increasing will yield the complement of the support of $\underline{F}$, and, together with c (to determine whether there is any mass at zero), the complement of the support of F. Figure 1 (b) provides an illustration. It is the graph of (1.2) corresponding to the density in (a).
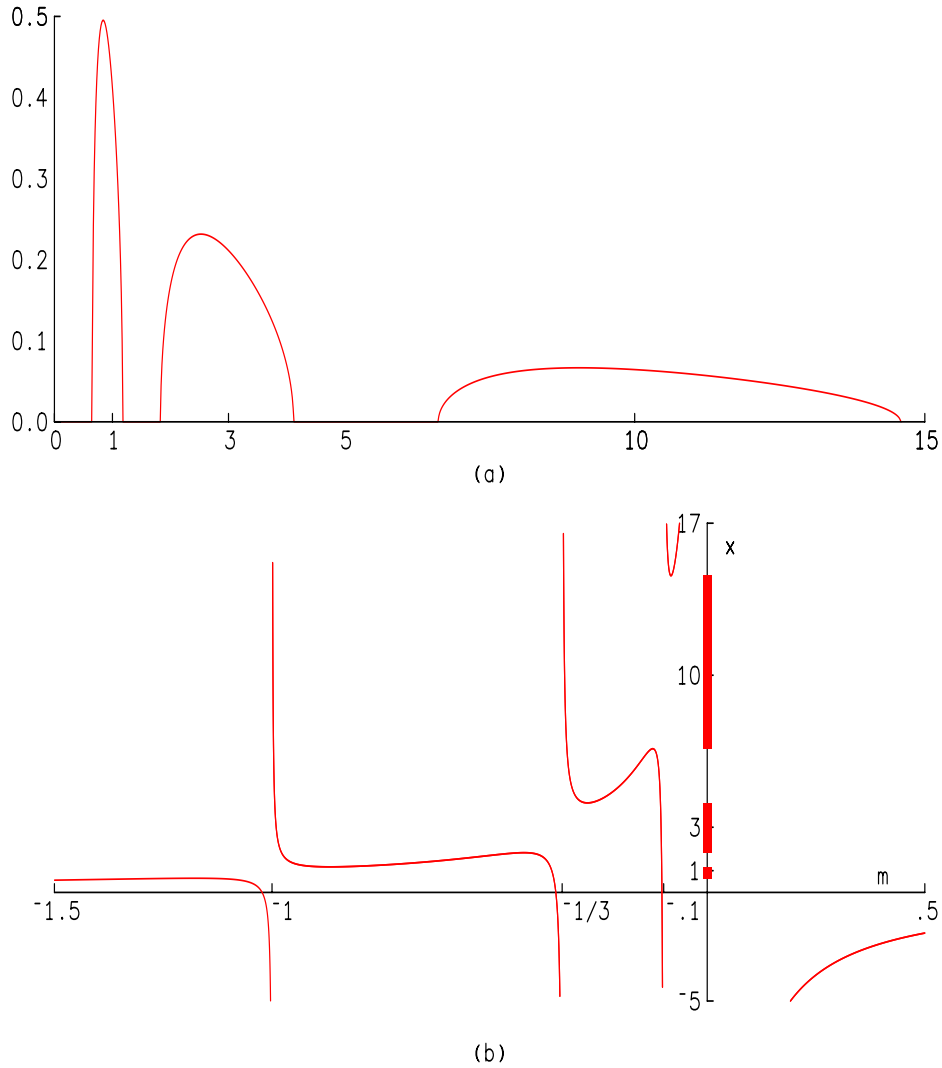
2

FIG 1. (a) *Graph of the limiting density when* $c = .1$ *and* H *places mass* $.2$, $.4$, *and* $.4$ *at,* *respectively,* 1, 3, *and* 10. (b) *The graph of* $x = -m^{-1} + c \int t(1+tm)^{-1}dH(t)$ *corresponding* *to* (a). *The bold lines on the vertical axis indicate the support of the density, the set in* $\mathbb{R}^+$ *remaining after removing intervals where the graph is increasing. Using the fact that the* *density at* $x \in \mathbb{R}^+$ *is equal to* $(c\pi)^{-1}$ *times the imaginary part of* $m_{\underline{F}}(x)$ (*see* Silverstein *and* Choi (1995)), *the graph in* (a) *was created by applying Newton's method to* (1.2) *for* *values of* $z = x$ *in the support.*

For large n one would intuitively expect no eigenvalues to appear on a closed interval outside the support of F. This of course cannot be inferred from the limiting result on F^{B_n}. The two important cases when $T = I$ have been settled. Here the support of F lies on $[(1 - \sqrt{c})^2, (1 + \sqrt{c})^2]$, with the addition of zero when $c > 1$. When the entries

3

of X come from the upper left portion of a doubly infinite array of independent random variables having finite fourth moment, Yin, Bai, and Krishnaiah (1988), and Bai and Yin (1993) show, respectively, the largest eigenvalue of B_n converges $a.s.$ to $(1 + \sqrt{c})^2$, and the $\min(n, N)^{\text{th}}$ largest (which is the smallest eigenvalue when $c < 1$) converges $a.s.$ to $(1 - \sqrt{c})^2$ (we remark here that in Bai, Silverstein, and Yin (1988), it is proven that $\mathsf{E}|X_{11}|^4 < \infty$ is necessary for the former to hold).

Extensive computer simulations, performed in order to show the importance of the spectral limiting results to the detection problem in array signal processing (Silverstein and Combettes (1992)), resulted in no eigenvalues appearing where there is no mass in the limit. Under reasonably mild conditions, this paper will provide a proof of this phenomenon, again in the form of a limit theorem as $n \to \infty$.

It will be necessary to impose stronger conditions on the eigenvalues of T_n than simply weak convergence of F^{T_n} to H. For this, if we let $F^{c,H}$ denote $\underline{F}$ and $c_n = n/N$, then F^{c_n, H_n} is the "limiting" nonrandom d.f. associated with the "limiting" ratio c_n and d.f. H_n. As will be seen, the conditions on H_n are reflected in F^{c_n, H_n}.

Theorem 1.1. Assume

(a) X_{ij}, $i, j = 1, 2, \ldots$ are i.i.d. random variables in $\mathbb{C}$ with $\mathsf{E}X_{11} = 0$, $\mathsf{E}|X_{11}|^2 = 1$, and $\mathsf{E}|X_{11}|^4 < \infty$.

(b) $N = N(n)$ with $c_n = n/N \to c > 0$ as $n \to \infty$.

(c) For each n $T = T_n$ is $n \times n$ Hermitian nonnegative definite satisfying $H_n \equiv F^{T_n} \xrightarrow{\mathcal{D}} H$, a p.d.f.

(d) $\|T_n\|$, the spectral norm of T_n is bounded in n.

(e) $B_n = (1/N)T_n^{1/2} X_n X_n^* T_n^{1/2}$, $T_n^{1/2}$ any Hermitian square root of T_n, $\underline{B}_n = (1/N)X_n^* T_n X_n$, where $X = X_n = (X_{ij})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, N$.

(f) Interval $[a, b]$ with $a > 0$ lies outside the support of $F^{c,H}$ and F^{c_n, H_n} for all large n

Then $\mathsf{P}($ no eigenvalue of B_n appears in $[a, b]$ for all large n $) = 1$.

Using the results on the extreme eigenvalues of $(1/N)XX^*$ we see that the interval can also be unbounded. In particular we have

Corollary. If $\|T_n\|$ converges to the largest number in the support of H, then $\|B_n\|$ converges $a.s.$ to the largest number in the support of F. If the smallest eigenvalue of T_n converges to the smallest number in the support of H, then $c < 1$ ($c > 1$) implies the smallest eigenvalue of B_n ($\underline{B}_n$) converges to the smallest number in the support of F ($\underline{F}$).

Theorem 1.1 is proven by showing the convergence of Stieltjes transforms at an appropriate rate, uniform with respect to the real part of z over certain intervals, while the imaginary part of z converges to zero. Besides relying on standard results on matrices, the proof requires well-known bounds on moments of martingale difference sequences, as

4

well as an extension of Rosenthal's inequality to random quadratic forms. The proof of the latter will be given in the appendix. Statements of most of the mathematical tools needed will be given in the next section. Section 3 establishes a rate of convergence of $F^{\underline{B}_n}$, needed in proving the convergence of the Stieltjes transforms. The latter will be broken down into two parts (section 4 and 5), while section 6 completes the proof.

It is mentioned here that Theorem 1.1 is actually only part of the important phenomena observed in simulations. It can be shown that on any interval J_H with endpoints outside the support of H, there corresponds for c sufficiently small, an interval $J_{F,c}$ with endpoints being boundary points of the support of F satisfying $F(J_{F,c}) = H(J_H)$. This should be viewed in the finite but large dimensional case as the eigenvalues of B_n being a "smoothed" deformation of the eigenvalues of T_n, continuous in the ratio of dimension to sample size. Simulations reveal that the number of eigenvalues of B_n appearing in J_{F_n,c_n} is exactly the same as the number of eigenvalues of T_n in J_{H_n}. The formulation of the conjecture naturally arising from this is simply

$$n(F_n(J_{F,c}) - F^{c_n,H_n}(J_{F,c})) \to 0 \quad a.s.$$

Its truth is currently being investigated.

2. Mathematical tools

We list in this section results needed to prove Theorem 1.1. Throughout the rest of the paper constants appearing in inequalities are represented by K and occasionally subscripted with variables they depend on. They are nonrandom and may take on different values from one appearance to the next.

Referenced results below concerning moments of sums of complex random variables were originally proven for real variables. Extension to the complex case is straightforward.

Lemma 2.1 (Burkholder (1973)). Let $\{X_k\}$ be a complex martingale difference sequence with respect to the increasing σ-field $\{\mathcal{F}_k\}$. Then for $p \geq 2$

$$\mathsf{E}\left|\sum X_k\right|^p \leq K_p\left(\mathsf{E}\left(\sum \mathsf{E}(|X_k|^2|\mathcal{F}_{k-1})\right)^{p/2} + \mathsf{E}\sum |X_k|^p\right).$$

Lemma 2.2 (Burkholder (1973)). With $\{X_k\}$ as above, we have, for $p > 1$

$$\mathsf{E}\left|\sum X_k\right|^p \leq K_p\mathsf{E}\left(\sum |X_k|^2\right)^{p/2}.$$

Lemma 2.3 (Rosenthal (1970)) If $\{X_k\}$ are independent non-negative, then for $p \geq 1$

$$\mathsf{E}\left(\sum X_k\right)^p \leq K_p\left(\left(\sum \mathsf{E}X_k\right)^p + \sum \mathsf{E}X_k^p\right).$$

5

Lemma 2.4 (Dilworth (1993)). With $\{\mathcal{F}_k\}$ as above, $\{X_k\}_{k \geq 1}$ a sequence of integrable random variables, and $1 \leq q \leq p < \infty$ we have

$$\mathsf{E}\left(\sum_{k=1}^{\infty} |\mathsf{E}(X_k|\mathcal{F}_k)|^q\right)^{p/q} \leq \left(\frac{p}{q}\right)^{p/q} \mathsf{E}\left(\sum_{k=1}^{\infty} |X_k|^q\right)^{p/q}.$$

The following lemma is found in most probability textbooks.

Lemma 2.5. (Kolmogorov's inequality for submartingales). If $X_1, \ldots, X_m$ is a submartingale, then for any $\alpha > 0$

$$\mathsf{P}(\max_{k \leq m} X_k \geq \alpha) \leq \frac{1}{\alpha}\mathsf{E}(|X_m|).$$

The next one has a straightforward proof.

Lemma 2.6. If for all $t > 0$, $P(|X| > t)t^p \leq K$ for some positive p, then for any positive $q < p$

$$E|X|^q \leq K^{q/p}\left(\frac{p}{p-q}\right).$$

Lemma 2.7 (proof in appendix). For $X = (X_1, \ldots, X_n)^T$ i.i.d. standardized (complex) entries, C $n \times n$ matrix (complex) we have for any $p \geq 2$

$$\mathsf{E}|X^*CX - \operatorname{tr}C|^p \leq K_p\left(\left(\mathsf{E}|X_1|^4\operatorname{tr}CC^*\right)^{p/2} + \mathsf{E}|X_1|^{2p}\operatorname{tr}(CC^*)^{p/2}\right).$$

Lemma 2.8 (Corollary 7.3.8 of Horn and Johnson (1985)). For $r \times s$ matrices A and B with respective singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q$, $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_q$, where $q = \min(r, s)$ we have

$$|\sigma_k - \tau_k| \leq \|B - A\| \quad \text{for all } k = 1, 2, \ldots, q.$$

Lemma 2.9 ((3.3.41) of Horn and Johnson (1991)). For $n \times n$ Hermitian $A = (a_{ij})$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, and convex f we have

$$\sum_{i=1}^{n} f(a_{ii}) \leq \sum_{i=1}^{n} f(\lambda_i).$$

Lemma 2.10 (Lemma 2.6 of Silverstein and Bai (1995)). Let $z \in \mathbb{C}^+$ with $v = Im\,z$, A and B $n \times n$ with B Hermitian, and $r \in \mathbb{C}^n$. Then

$$\left|\operatorname{tr}\left((B - zI)^{-1} - (B + rr^* - zI)^{-1}\right)A\right| = \left|\frac{r^*(B - zI)^{-1}A(B - zI)^{-1}r}{1 + r^*(B - zI)^{-1}r}\right| \leq \frac{\|A\|}{v}.$$

6

Lemma 2.11 (Lemma 2.3 of Silverstein (1995)). For $z = x + iv \in \mathbb{C}^+$ let $m_1(z)$, $m_2(z)$ be Stieltjes transforms of any two p.d.f.'s, A and B $n \times n$ with A Hermitian non-negative definite, and $r \in \mathbb{C}^n$. Then

a) $$\|(m_1(z)A + I)^{-1}\| \leq \max(4\|A\|/v, 2)$$

b) $|\mathrm{tr}\, B((m_1(z)A + I)^{-1} - (m_2(z)A + I)^{-1})| \leq |m_2(z) - m_1(z)|n\|B\|\,\|A\|(\max(4\|A\|/v, 2))^2$

c) $|r^* B(m_1(z)A + I)^{-1}r - r^* B(m_2(z)A + I)^{-1}r|$
$$\leq |m_2(z) - m_1(z)|\|r\|^2\|B\|\|A\|(\max(4\|A\|/v, 2))^2$$

($\|r\|$ denoting Euclidean norm on r).

Lemma 2.12 (Lemma 2.4 of Silverstein and Bai (1995)). For $n \times n$ Hermitian A and B

$$\|F^A - F^B\| \leq \frac{1}{n}\mathrm{rank}(A - B),$$

$\|\cdot\|$ here denoting sup norm on functions.

Basic properties on matrices will be used throughout the paper, the two most common being: $\mathrm{tr}\, AB \leq \|A\|\mathrm{tr}\, B$ for Hermitian nonnegative definite A and B, and for A $n \times n$ and $r \in \mathbb{C}^n$, for which both A and $A + rr^*$ are invertible

$$r^*(A + rr^*)^{-1} = \frac{1}{(1 + r^* A^{-1} r)}r^* A^{-1}.$$

At one point in Section 3 the two-dimensional Stieltjes transform is needed. Its definition and relevant properties are given here. For a p.d.f. $F(x, y)$ defined on $\mathbb{R}^2$ it is defined as

$$m(z_1, z_2) = \int \frac{1}{(x - z_1)(y - z_2)}dF(x, y)$$

for all $z_1 = x_1 + iv_1$, $z_2 = x_2 + iv_2$ $v_1 \neq 0$, $v_2 \neq 0$. Due to the inversion formula

$$F([a, b] \times [c, d]) = -\frac{1}{\pi^2}\lim_{v \downarrow 0} \int_{[a,b] \times [c,d]} m(z_1, z_2) - m(\bar{z}_1, z_2) - m(z_1, \bar{z}_2) + m(\bar{z}_1, \bar{z}_2)dx_1 dx_2$$

$v_1 = v_2 = v$, whenever $F(\partial([a, b] \times [c, d])) = 0$, weak convergence of p.d.f.'s on $\mathbb{R}^2$ is assured once convergence of their Stieltjes transforms is verified on a countable collection of points (z_1, z_2) dense in some open set in $\mathbb{C}^2$.

3. A rate on $F^{\underline{B}_n}$

We begin by simplifying our assumptions.

Because of assumption (d) in Theorem 1.1 we can assume $\|T_n\| \le 1$.

For $C > 0$ let $Y_{ij} = X_{ij}I_{[|X_{ij}|\le C]} - \mathsf{E}X_{ij}I_{[|X_{ij}|\le C]}$, $Y = (Y_{ij})$ and $\tilde{B}_n = (1/N)T_n^{1/2}Y_nY_n^*T_n^{1/2}$. Denote the eigenvalues of B_n and $\tilde{B}_n$ by λ_k and $\tilde{\lambda}_k$ (in decreasing order). Since these are the squares of the kth largest singular values of $(1/\sqrt{N})T_n^{1/2}X_n$ and $(1/\sqrt{N})T_n^{1/2}Y_n$ (respectively), we find using Lemma 2.8

$$\max_{k \le n}|\lambda_k^{1/2} - \tilde{\lambda}_k^{1/2}| \le (1/\sqrt{N})\|X_n - Y_n\|.$$

Since $X_{ij} - Y_{ij} = X_{ij}I_{[|X_{ij}|>C]} - \mathsf{E}X_{ij}I_{[|X_{ij}|>C]}$, from Yin, Bai, and Krishnaiah (1988) we have with probability one

$$\limsup_{n\to\infty}\max_{k\le n}|\lambda_k^{1/2} - \tilde{\lambda}_k^{1/2}| \le (1 + \sqrt{c})\mathsf{E}^{1/2}|X_{11}|^2I_{[|X_{11}|>C]}.$$

Because of assumption (a) we can make the above bound arbitrarily small by choosing C sufficiently large. Thus, in proving Theorem 1.1 it is enough to consider the case where the underlying variables are uniformly bounded.

In this case it is proven in Yin, Bai, and Krishnaiah (1988) that there exists a sequence $\{k_n\}$ satisfying $k_n/\log n \to \infty$ such that for any $\eta > (1 + \sqrt{c})^2$

$$\mathsf{E}\|(1/N)X_nX_n^*\|^{k_n} \le \eta^{k_n}$$

for all n sufficiently large. It follows then that $\lambda_{\max}$, the largest eigenvalue of B_n, satisfies.

$$P(\lambda_{\max} \ge K) = o(N^{-\ell}), \tag{3.1}$$

for any $K > (1 + \sqrt{c})^2$ and any positive ℓ.

Also, since $\mathsf{tr}\,(CC^*)^{p/2} \le (\mathsf{tr}\,CC^*)^{p/2}$, we get from Lemma 2.7 when X_1 is bounded

$$\mathsf{E}|X_{\cdot 1}^*CX_{\cdot 1} - \mathsf{tr}\,C|^p \le K_p(\mathsf{tr}\,CC^*)^{p/2} \tag{3.2}$$

where K_p also depends on the distribution of $X_{\cdot 1}$. From (3.2) we easily get

$$\mathsf{E}|X_{\cdot 1}^*CX_{\cdot 1}|^p \le K_p((\mathsf{tr}\,CC^*)^{p/2} + |\mathsf{tr}\,C|^p). \tag{3.3}$$

Throughout the paper, variable $z = x + iv$ will be the argument of any Stieltjes transform. Let $m_n = m_{F^{B_n}}$ and $\underline{m}_n = m_{F^{\underline{B}_n}}$. For $j = 1, 2, \ldots, N$, let $q_j = (1/\sqrt{n})X_{\cdot j}$ ($X_{\cdot j}$ denoting the jth column of X), $r_j = (1/\sqrt{N})T_n^{1/2}X_{\cdot j}$, and $B_{(j)} = B_{(j)}^n = B_n - r_jr_j^*$.

8

In Silverstein (1995) the formula

$$\underline{m}_n(z) = -\frac{1}{N}\sum_{j=1}^{N}\frac{1}{z(1 + r_j^*(B_{(j)} - zI)^{-1}r_j)}$$

is derived. It is easy to verify

$$Im\, r_j^*((1/z)B_{(j)} - I)^{-1}r_j \geq 0.$$

Therefore, for each j

$$\frac{1}{|z(1 + r_j^*(B_{(j)} - zI)^{-1}r_j)|} \leq \frac{1}{v}, \tag{3.4}$$

It is also shown in Silverstein (1995) that

$$\frac{1}{n}\mathsf{tr}\left(-z\underline{m}_n(z)T_n - zI\right)^{-1} - m_n(z) \equiv w_n(z) = \frac{1}{N}\sum_{j=1}^{N}\frac{-1}{z(1 + r_j^*(B_{(j)} - zI)^{-1}r_j)}d_j \tag{3.5}$$

where

$$d_j = q_j^*T_n^{1/2}(B_{(j)} - zI)^{-1}(\underline{m}_n(z)T_n + I)^{-1}T_n^{1/2}q_j - \frac{1}{n}\mathsf{tr}\left(\underline{m}_n(z)T_n + I\right)^{-1}T_n(B_n - zI)^{-1}.$$

The next task is to prove for $v = v_n \geq N^{-1/17}$ and for any subsets $S_n \subset [0, \infty)$ containing at most n elements the almost sure convergence of

$$\max_{x \in S_n}\frac{|w_n(z)|}{v_n^5}$$

to zero. Let $\underline{m}_{(j)}(z) = -\frac{(1 - c_n)}{z} + c_n m_{F^{B_{(j)}}}(z)$. From Lemma 2.10 we have

$$\max_{j \leq N}|\underline{m}_n(z) - \underline{m}_{(j)}(z)| \leq \frac{1}{Nv}. \tag{3.6}$$

Moreover, it is easy to verify that $\underline{m}_{(j)}(z)$ is the Stieltjes transform of a p.d.f., so that $|\underline{m}_{(j)}(z)| \leq v^{-1}$

Write for each $j \leq N$ $d_j = d_j^1 + d_j^2 + d_j^3 + d_j^4$ where

$$d_j^1 = q_j^*T_n^{1/2}(B_{(j)} - zI)^{-1}(\underline{m}_n(z)T_n + I)^{-1}T_n^{1/2}q_j - q_j^*T_n^{1/2}(B_{(j)} - zI)^{-1}(\underline{m}_{(j)}(z)T_n + I)^{-1}T_n^{1/2}q_j$$

$$d_j^2 = q_j^*T_n^{1/2}(B_{(j)} - zI)^{-1}(\underline{m}_{(j)}(z)T_n + I)^{-1}T_n^{1/2}q_j - \frac{1}{n}\mathsf{tr}\left(\underline{m}_{(j)}(z)T_n + I\right)^{-1}T_n(B_{(j)} - zI)^{-1}$$

$$d_j^3 = \frac{1}{n}\mathsf{tr}\left(\underline{m}_{(j)}(z)T_n + I\right)^{-1}T_n(B_{(j)} - zI)^{-1} - \frac{1}{n}\mathsf{tr}\left(\underline{m}_{(j)}(z)T_n + I\right)^{-1}T_n(B_n - zI)^{-1}.$$

9

and

$$d_j^4 = \tfrac{1}{n}\mathsf{tr}\,(\underline{m}_{(j)}(z)T_n + I)^{-1}T_n(B_n - zI)^{-1} - \tfrac{1}{n}\mathsf{tr}\,(\underline{m}_n(z)T_n + I)^{-1}T_n(B_n - zI)^{-1}.$$

In view of (3.4), it is sufficient to show the *a.s.* convergence of

$$\max_{j \le N, x \in S_n} \frac{|d_j^i|}{v^6} \tag{3.7}$$

to zero for $i = 1, 2, 3, 4$.

Using $\|(A - zI)^{-1}\| \le 1/v$ for any Hermitian matrix A we get from Lemma 2.11 c) and (3.6)

$$|d_j^1| \le 16\frac{\|X_{\cdot j}\|^2}{n}\frac{1}{Nv^4}.$$

Using (3.2) it follows that for any $\epsilon > 0$, $p \ge 2$, and all n sufficiently large

$$\mathsf{P}\left(\max_{j \le N, x \in S_n} \frac{|d_j^1|}{v^6} > \epsilon\right) \le n\mathsf{P}\left(\max_{j \le N}\left|\frac{\|X_{\cdot j}\|^2}{n} - 1\right|\frac{16}{Nv^{10}} > \epsilon/2\right)$$

$$\le K_p \frac{nN}{(Nv^{10})^p}\epsilon^{-p}n^{-p/2},$$

so $(3.7) \xrightarrow{a.s.} 0$ when $i = 1$ and for any $v_n \in (N^{-1/10}, 1]$.

Using Lemma 2.10 and Lemma 2.11 a) we find

$$v^{-6}|d_j^3| \le \frac{4}{nv^8},$$

so that $(3.7) \xrightarrow{a.s.} 0$ for $i = 3$ and for any $v_n = N^{-\delta}$ with $\delta \in [0, 1/8)$.

We get from Lemma 2.11 b) and (3.6)

$$v^{-6}|d_j^4| \le 16\frac{1}{Nv^{10}},$$

so that $(3.7) \xrightarrow{a.s.} 0$ for $i = 4$, and for any $\delta \in [0, 1/10)$.

Using (3.2) we find for any $p \ge 2$

$\mathsf{E}|v^{-6}d_j^2|^p$

$\le K_p\dfrac{1}{v^{6p}n^p}(\mathsf{tr}\,T_n^{1/2}(B_{(j)}-zI)^{-1}(\underline{m}_{(j)}(z)T_n+I)^{-1}T_n(\overline{\underline{m}}_{(j)}(z)T_n+I)^{-1}(B_{(j)}-\overline{z}I)^{-1}T_n^{1/2})^{p/2}$

$= K_p\dfrac{1}{v^{6p}n^p}(\mathsf{tr}\,(\underline{m}_{(j)}(z)T_n + I)^{-1}T_n(\overline{\underline{m}}_{(j)}(z)T_n + I)^{-1}(B_{(j)} - \overline{z}I)^{-1}T_n(B_{(j)} - zI)^{-1})^{p/2}$

$\le\ (\text{using Lemma 2.11 a)})\ K_p\dfrac{1}{v^{6p}n^p}\dfrac{1}{v^{2\cdot p/2}}(\mathsf{tr}\,(B_{(j)} - \overline{z}I)^{-1}T_n(B_{(j)} - zI)^{-1})^{p/2}$

10

$$= K_p \frac{1}{(nv^7)^p} (\operatorname{tr} T_n (B_{(j)} - zI)^{-1} (B_{(j)} - \overline{z}I)^{-1})^{p/2} \le K_p \frac{1}{(nv^7)^p} (n/v^2)^{p/2}$$

$$= K_p \frac{1}{(n^{1/2} v^8)^p}.$$

We have then for any $\epsilon > 0$ and $p \ge 2$

$$\mathsf{P}(\max_{j \le N, x \in S_n} |v^{-6} d_j^2| > \epsilon) \le K_p \frac{1}{\epsilon^p} \frac{nN}{(n^{1/2} v^8)^p}.$$

Thus, $\max_{x \in S_n} |w_n(z)| v^{-5} \xrightarrow{a.s.} 0$ for any nonnegative $\delta \le 1/17$ since we have shown for any positive ℓ, we have for all p sufficiently large and for all $\epsilon > 0$

$$\mathsf{P}(\max_{x \in S_n} |w_n(z)| v_n^{-5} > \epsilon) \le K_p \epsilon^{-p} n^{-\ell}.$$

Moreover, for the sequence $\{\mu_n\}$ with $\mu_n = N^{1/68}$ we have for any $v_n = N^{-\delta}$ with $\delta \le 1/17$

$$\mathsf{P}(\mu_n \max_{x \in S_n} |w_n(z)| v_n^{-5} > \epsilon) \le K_p \epsilon^{-p} n^{-\ell}. \tag{3.8}$$

We now rewrite w_n totally in terms of $\underline{m}_n$. With $H_n \equiv F^T$, and using the identity

$$\underline{m}_n(z) = -\frac{(1 - c_n)}{z} + c_n m_n(z)$$

we have $w_n =$

$$\frac{1}{c_n} \left(-\frac{c_n}{z} \int \frac{dH_n(t)}{1 + t\underline{m}_n} - \underline{m}_n - \frac{(1 - c_n)}{z} \right) = \frac{\underline{m}_n}{c_n z} \left(-\frac{c_n}{\underline{m}_n} \int \frac{dH_n(t)}{1 + t\underline{m}_n} - z - \frac{(1 - c_n)}{\underline{m}_n} \right)$$

$$= \frac{\underline{m}_n}{c_n z} \left(-z - \frac{1}{\underline{m}_n} + c_n \int \frac{t \, dH_n(t)}{1 + t\underline{m}_n} \right).$$

Let

$$\omega = -z - \frac{1}{\underline{m}_n} + c_n \int \frac{t \, dH_n(t)}{1 + t\underline{m}_n}.$$

Then $\omega = w_n z c_n / \underline{m}_n$.

Returning now to F^{c_n, H_n} and $F^{c, H}$, let $\underline{m}_n^0 = m_{F^{c_n, H_n}}$ and $\underline{m}^0 = m_{F^{c, H}}$. Then $\underline{m}^0$ solves (1.1), its inverse is given by (1.2),

$$\underline{m}_n^0 = \frac{1}{-z + c_n \int \frac{t \, dH_n(t)}{1 + t\underline{m}_n^0}}, \tag{3.9}$$

and the inverse of $\underline{m}_n^0$, denoted z_n^0, is given by

$$z_n^0(\underline{m}) = -\frac{1}{\underline{m}} + c_n \int \frac{t \, dH_n(t)}{1 + t\underline{m}}. \tag{3.10}$$

11

From (3.10) and the inversion formula for Stieltjes transforms it is obvious that $F^{c_n, H_n} \xrightarrow{\mathcal{D}} F^{c,H}$, as $n \to \infty$. Therefore, from assumption (f) an $\underline{\epsilon} > 0$ exists for which $[a - 2\underline{\epsilon}, b + 2\underline{\epsilon}]$ also satisfies (f). This interval will stay uniformly bounded away from the boundary of the support of F^{c_n, H_n} for all large n, so that for these n both $\sup_{x \in [a-2\underline{\epsilon}, b+2\underline{\epsilon}]} \frac{d}{dx} \underline{m}_n^0(x)$ is bounded and $-1/\underline{m}_n^0(x)$ for $x \in [a - 2\underline{\epsilon}, b + 2\underline{\epsilon}]$ stays uniformly away from the support of H_n. Therefore for all n sufficiently large

$$\sup_{x \in [a-2\underline{\epsilon}, b+2\underline{\epsilon}]} \left(\frac{d}{dx} \underline{m}_n^0(x) \right) \int \frac{t^2 \, dH_n(t)}{(1 + t\underline{m}_n^0(x))^2} \leq K. \tag{3.11}$$

Let $a' = a - \underline{\epsilon}$, $b' = b + \underline{\epsilon}$. On either $(-\infty, a']$ or $[b', \infty)$, each collection of functions in λ, $\{(\lambda - x)^{-1} : x \in [a, b]\}$, $\{(\lambda - x)^{-2} : x \in [a, b]\}$, form a uniformly bounded, equicontinuous family. It is straightforward, then, to show

$$\lim_{n \to \infty} \sup_{x \in [a,b]} |\underline{m}_n^0(x) - \underline{m}^0(x)| = 0, \tag{3.12}$$

and

$$\lim_{n \to \infty} \sup_{x \in [a,b]} \left| \frac{d}{dx} \underline{m}_n^0(x) - \frac{d}{dx} \underline{m}^0(x) \right| = 0 \tag{3.13}$$

(see, for example, p. 17, problem 8 of Billingsley (1968)). Since for all $x \in [a, b]$, $\lambda \in [a', b']^c$, and positive v

$$\left| \frac{1}{\lambda - (x + iv)} - \frac{1}{\lambda - x} \right| < \frac{v}{\underline{\epsilon}^2}$$

we have for any sequence of positive v_n converging to 0

$$\lim_{n \to \infty} \sup_{x \in [a,b]} |\underline{m}_n^0(x + iv_n) - \underline{m}_n^0(x)| = 0. \tag{3.14}$$

Similarly

$$\lim_{n \to \infty} \sup_{x \in [a,b]} \left| \frac{Im \, \underline{m}_n^0(x + iv_n)}{v_n} - \frac{d}{dx} \underline{m}_n^0(x) \right| = 0. \tag{3.15}$$

Expressions (3.11), (3.12), (3.14), and (3.15) will be needed in the latter part of Section 5.

Let $\underline{m}_2^0 = Im \, \underline{m}_n^0$. We have then from (3.9)

$$\underline{m}_2^0 = \frac{v_n + \underline{m}_2^0 c_n \int \frac{t^2 \, dH_n(t)}{|1 + t\underline{m}_n^0|^2}}{\left| -z + c_n \int \frac{t \, dH_n(t)}{1 + t\underline{m}_n^0} \right|^2}. \tag{3.16}$$

For any real x, by Lemma 2.11 a),

$$\underline{m}_2^0 c_n \int \frac{t^2 \, dH_n(t)}{|1 + t\underline{m}_n^0|^2} = c_n Im\left(\int \frac{t \, dH_n(t)}{1 + t\underline{m}_n^0}\right)$$

$$\leq c_n \|T(I + T\underline{m}_n^0)^{-1}\| \leq 4c_n/v_n.$$

It follows that

$$\left(\frac{\underline{m}_2^0 c_n \int \frac{t^2 \, dH_n(t)}{|1+t\underline{m}_n^0|^2}}{v_n + \underline{m}_2^0 c_n \int \frac{t^2 \, dH_n(t)}{|1+t\underline{m}_n^0|^2}}\right)^{1/2} < 1 - Kv_n^2, \tag{3.17}$$

for some positive constant K.

Let $\underline{m}_n = \underline{m}_1 + i\underline{m}_2$, where $\underline{m}_1 = Re\,\underline{m}_n$, $\underline{m}_2 = Im\,\underline{m}_n$. We have $\underline{m}_n$ satisfying

$$\underline{m}_n = \frac{1}{-z + c_n \int \frac{t \, dH_n(t)}{1+t\underline{m}_n} - \omega}, \tag{3.18}$$

and

$$\underline{m}_2 = \frac{v_n + \underline{m}_2 c_n \int \frac{t^2 \, dH_n(t)}{|1+t\underline{m}_n|^2} + Im\,\omega}{\left|-z + c_n \int \frac{t \, dH_n(t)}{1+t\underline{m}_n} - \omega\right|^2}. \tag{3.19}$$

From (3.9) and (3.18) we get

$$\underline{m}_n - \underline{m}_n^0 = \frac{(\underline{m}_n - \underline{m}_n^0)c_n \int \frac{t^2 \, dH_n(t)}{(1+t\underline{m}_n)(1+t\underline{m}_n^0)}}{\left(-z + c_n \int \frac{t \, dH_n(t)}{1+t\underline{m}_n} - \omega\right)\left(-z + c_n \int \frac{t \, dH_n(t)}{1+t\underline{m}_n^0}\right)}$$

$$+ \underline{m}_n \underline{m}_n^0 \omega. \tag{3.20}$$

From Cauchy-Schwarz, (3.16), (3.17) and (3.19) we get, when $|Im\,\omega/v_n| < 1$

$$\left| c_n \frac{\int \frac{t^2 \, dH_n(t)}{(1+t\underline{m}_n)(1+t\underline{m}_n^0)}}{\left(-z + c_n \int \frac{t \, dH_n(t)}{1+t\underline{m}_n} - \omega\right)\left(-z + c_n \int \frac{t \, dH_n(t)}{1+t\underline{m}_n^0}\right)} \right|$$

$$\leq \left(c_n \frac{\int \frac{t^2 \, dH_n(t)}{|1+t\underline{m}_n|^2}}{\left|-z + c_n \int \frac{t \, dH_n(t)}{1+t\underline{m}_n} - \omega\right|^2} \right)^{1/2} \left(c_n \frac{\int \frac{t^2 \, dH_n(t)}{|1+t\underline{m}_n^0|^2}}{\left|-z + c_n \int \frac{t \, dH_n(t)}{1+t\underline{m}_n^0}\right|^2} \right)^{1/2}$$

$$= \left(\frac{\underline{m}_2 c_n \int \frac{t^2 \, dH_n(t)}{|1+t\underline{m}_n|^2}}{v_n + \underline{m}_2 c_n \int \frac{t^2 \, dH_n(t)}{|1+t\underline{m}_n|^2} + Im\,\omega} \right)^{1/2} \left(\frac{\underline{m}_2^0 c_n \int \frac{t^2 \, dH_n(t)}{|1+t\underline{m}_n^0|^2}}{v_n + \underline{m}_2^0 c_n \int \frac{t^2 \, dH_n(t)}{|1+t\underline{m}_n^0|^2}} \right)^{1/2}$$

13

$$\leq \left(\frac{\underline{m}_2^0 c_n \int \frac{t^2\, dH_n(t)}{|1+t\underline{m}_n^0|^2}}{v_n + \underline{m}_2^0 c_n \int \frac{t^2\, dH_n(t)}{|1+t\underline{m}_n^0|^2}} \right)^{1/2}$$

$$\leq 1 - \underline{K} v_n^2. \tag{3.21}$$

We claim that on the set $\{\lambda_{\max} \leq K_1\}$, where $K_1 > (1+\sqrt{c})^2$, for all n sufficiently large, $|\underline{m}_n| \geq \frac{1}{2}\mu_n^{-1} v_n$ whenever $|x| \leq \mu_n v_n^{-1}$. Indeed, when $x \leq -v_n$ or $x \geq \lambda_{\max} + v_n$

$$|\underline{m}_n| \geq |Re\,\underline{m}_n| \geq \frac{K_1 + \mu_n v_n^{-1}}{(K_1 + \mu_n v_n^{-1})^2 + v_n^2} \geq \frac{1}{2\mu_n v_n^{-1}}$$

for n large. When $-v_n < x < \lambda_{\max} + v_n$

$$|\underline{m}_n| \geq |Im\,\underline{m}_n| \geq \frac{v_n}{(K_1 + v_n)^2 + v_n^2} \geq \mu_n^{-1} v_n$$

for n large. Thus the claim is proven.

Therefore, when $|x| \leq \mu_n v_n^{-1}$, on the set $\{|w_n| \leq v_n^4\} \cap \{\lambda_{\max} \leq K_1\}$ we have for n large $|z| \leq 2\mu_n v_n^{-1}$ and

$$|Im(\omega)| \leq |c_n z w_n / \underline{m}_n| \leq K\mu_n^2 v_n^{-2} |w_n| < v_n.$$

Therefore, by (3.20) and (3.21), we have

$$|\underline{m}_n - \underline{m}_n^0| \leq \underline{K}^{-1} v_n^{-2} |\underline{m}_n \underline{m}_n^0 \omega|$$

$$= \underline{K}^{-1} v_n^{-2} |c_n z \underline{m}_n^0 w_n| \leq K' v_n^{-4} \mu_n |w_n|.$$

It is easy to verify that for n large, when either $|x| > \mu_n v_n^{-1}$, $|w_n| > v_n^4$, or $\lambda_{\max} > K_1$

$$|\underline{m}_n - \underline{m}_n^0| \leq 3\mu_n^{-1} v_n + 2v_n^{-1}(I_{[|w_n|>v_n^4]} + I_{[\lambda_{\max}>K_1]}).$$

Therefore, for n large, we have

$$\max_{x \in S_n} v_n^{-1} |\underline{m}_n(z) - \underline{m}_n^0| \leq K'\mu_n \max_{x \in S_n} |w_n| v_n^{-5}$$

$$+3\mu_n^{-1} + 2v_n^{-2} \max_{x \in S_n} \left(I_{[|w_n|>v_n^4]} + I_{[\lambda_{\max}>K_1]} \right).$$

Therefore, from (3.1) and (3.8) we find for any positive ϵ and ℓ

$$\mathsf{P}(v_n^{-1} \max_{x \in S_n} |\underline{m}_n(z) - \underline{m}_n^0| > \epsilon) \leq K_p \epsilon^{-p} n^{-\ell} \tag{3.22}$$

for all p sufficiently large, whenever $\delta \leq 1/17$.

We now assume the n elements of S_n to be equally spaced between $-\sqrt{n}$ and $\sqrt{n}$. Since for $|x_1 - x_2| \le 2n^{-1/2}$,

$$|\underline{m}_n(x_1 + iv_n) - \underline{m}_n(x_2 + iv_n)| \le 2n^{-1/2}v_n^{-2}$$

$$|\underline{m}_n^0(x_1 + iv_n) - \underline{m}_n^0(x_2 + iv_n)| \le 2n^{-1/2}v_n^{-2},$$

and when $|x| \ge \sqrt{n}$, for n large

$$|\underline{m}_n(x + iv_n)| \le 2n^{-1/2} + v_n^{-1}I_{[\lambda_{\max} > K_1]}$$

$$|\underline{m}_n^0(x + iv_n)| \le 2n^{-1/2}$$

we conclude from (3.22) and (3.1), that for any positive ϵ and ℓ

$$\mathsf{P}(v_n^{-1} \sup_{x \in \mathbb{R}} |\underline{m}_n(x + iv_n) - \underline{m}_n^0(x + iv_n)| > \epsilon) \le K_p \epsilon^{-p} n^{-\ell} \qquad (3.23)$$

for all sufficiently large p, whenever $\delta \le 1/17$.

Let $\mathsf{E}_0(\cdot)$ denote expectation and $\mathsf{E}_k(\cdot)$ denote conditional expectation with respect to the σ-field generated by $r_1, \cdots, r_k$. Let $\ell, \ell' > 0$ be arbitrary. Choose $\underline{\ell} > \ell$, let p be suitably large so that (3.23) holds with ℓ replaced by $\underline{\ell}$, and $r \equiv \ell p/(\underline{\ell}\ell')$ is greater than 1. Since $\mathsf{E}_k(v_n^{-\ell'} \sup_{x \in \mathbb{R}} |\underline{m}_n(x + iv_n) - \underline{m}_n^0(x + iv_n)|^{\ell'})$, $k = 0, \ldots, N$ forms a martingale, it follows from Jensen's inequality, Lemmas 2.5, 2.6 and (3.23) that for any positive ϵ

$$\mathsf{P}(\max_{k \le N} \mathsf{E}_k(v_n^{-\ell'} \sup_{x \in \mathbb{R}} |\underline{m}_n(x + iv_n) - \underline{m}_n^0(x + iv_n)|^{\ell'}) > \epsilon)$$

$$\le \epsilon^{-r} \mathsf{E}(v_n^{-r\ell'} \sup_{x \in \mathbb{R}} |\underline{m}_n(x + iv_n) - \underline{m}_n^0(x + iv_n)|^{r\ell'}) \le \epsilon^{-r} K_p^{\ell/\underline{\ell}} \frac{\underline{\ell}}{\underline{\ell} - \ell} n^{-\ell}$$

whenever $\delta \le 1/17$. In particular, we have for $\delta \le 1/17$

$$\lim_{n \to \infty} \max_{k \le N} \frac{\mathsf{E}_k(\sup_{x \in \mathbb{R}} |\underline{m}_n(x + iv_n) - \underline{m}_n^0(x + iv_n)|^2)}{v_n^2} = 0 \qquad \text{with probability 1.} \quad (3.24).$$

Let $\lambda_1 \le \lambda_2 \cdots \le \lambda_N$ be the eigenvalues of $\underline{B}_n$ and write

$$\underline{m}_j = \underline{m}_j^{out} + \underline{m}_j^{in} \quad j = 1, 2$$

where

$$\underline{m}_2^{out}(x + iv) = \frac{1}{N} \sum_{\lambda_j \in [a', b']} \frac{v}{(x - \lambda_j)^2 + v^2},$$

15

$$\underline{m}_1^{out}(x + iv) = \frac{1}{N} \sum_{\lambda_j \in [a', b']} \frac{x - \lambda_j}{(x - \lambda_j)^2 + v^2}.$$

Define the sequence $\{G_m\}_{m=1}^{\infty}$ of functions on $\mathbb{R}^2$ by

$$G_{\sum_{j=1}^{n-1}(N(j)+1)+k}(x_1, x_2) = \mathsf{E}_k F^{\underline{B}_n}(x_1) F^{\underline{B}_n}(x_2),$$

for $k = 0, 1, \ldots, N(n)$. Clearly each G_m is a probability distribution function on $\mathbb{R}^2$, and when $m = \sum_{j=1}^{n-1}(N(j)+1)+k$, the two-dimensional Stieltjes transform, $m_m^{(G)}(x_1+iv_1, x_2+iv_2)$ of G_m is $\mathsf{E}_k \underline{m}_n(x_1 + iv_1) \underline{m}_n(x_2 + iv_2)$. Obviously, when $\delta = 0$, (3.24) implies that, with probability one $\sup_{x_1, x_2 \in \mathbb{R}} |m_m^{(G)}(x_1 + iv_1, x_2 + iv_2) - \underline{m}_n^0(x_1 + iv_1)\underline{m}_n^0(x_2 + iv_2)| \to 0$ as $m \to \infty$ for countably many (v_1, v_2) forming a dense subset of an open set in the first quadrant (bounded uniformly away from the two axes). We conclude that with probability one, $G_m(x_1, x_2)$ converges weakly to $F^{c,H}(x_1)F^{c,H}(x_2)$.

Since the integrands of

$$\int_{[a',b']^c \times [a,'b']^c} \frac{d\mathsf{E}_k F^{\underline{B}_n}(x_1) F^{\underline{B}_n}(x_2)}{((x - x_1)^2 + v^2)((x - x_2)^2 + v^2)} \quad \text{and} \quad \int_{[a',b']^c} \frac{d\mathsf{E}_k F^{\underline{B}_n}(x_1)}{(x - x_1)^2 + v^2}$$

on their respective domains are uniformly bounded and equicontinuous for $x \in [a, b]$, it follows as in (3.13) that

$$\max_k \sup_{x \in [a,b]} \mathsf{E}_k \left| \frac{m_2^{in}(x + iv)}{v} - \frac{d}{dx} \underline{m}^0(x) \right|^2 \xrightarrow{a.s.} 0, \tag{3.25}$$

for any $v = v_n \to 0$.

Therefore, from (3.24) and (3.25) we have

$$\max_{k \leq N} \sup_{x \in [a,b]} v_n^{-2} \mathsf{E}_k \left(\underline{m}_2^{out}(x + iv_n) \right)^2 \xrightarrow{a.s.} 0. \tag{3.26}$$

From (3.26) we can infer a bound on the number of eigenvalues in $[a, b]$. Notice $NF^{\underline{B}_n}(A)$ is the number of eigenvalues of $\underline{B}_n$ in the set A. Let e_n denote the left side of (3.26). For any $x \in [a, b]$

$$e_n \geq \frac{1}{N^2} \max_{k \leq N} \mathsf{E}_k \left(\sum_{\lambda_j \in [a,b] \cap [x - v_n, x + v_n]} \frac{1}{(x - \lambda_j)^2 + v_n^2} \right)^2$$

$$\geq \max_{k \leq N} \frac{N^2 \mathsf{E}_k \left(F^{\underline{B}_n}\{[a, b] \cap [x - v_n, x + v_n]\} \right)^2}{4v_n^4 N^2},$$

16

and since the number of intervals of length $2v_n$ needed to cover $[a,b]$ is $\lceil(b-a)/2v_n\rceil$, we find $\mathsf{E}_k\left(F^{\underline{B}_n}\{[a,b]\}\right)^2 \le (b-a)^2 v_n^2 e_n$. Therefore

$$\max_{k\le N} \mathsf{E}_k\left(F^{\underline{B}_n}\{[a,b]\}\right)^2 = o_{a.s.}(v_n^2) = o_{a.s.}(N^{-2/17}),$$

which implies

$$\max_{k\le N} \mathsf{E}_k(F^{\underline{B}_n}\{[a,b]\}) = o_{a.s.}(v_n) = o_{a.s.}(N^{-1/17}).$$

The above arguments apply to $[a',b']$ as well, so we also have

$$\max_{k\le N} \mathsf{E}_k\left(F^{\underline{B}_n}\{[a',b']\}\right)^2 = o_{a.s.}(v_n^2) = o_{a.s.}(N^{-2/17}) \tag{3.27}$$

and

$$\max_{k\le N} \mathsf{E}_k(F^{\underline{B}_n}\{[a',b']\}) = o_{a.s.}(v_n) = o_{a.s.}(N^{-1/17}). \tag{3.28}$$

4. Convergence of $m_n - \mathsf{E}m_n$

We now restrict $\delta = 1/68$, that is, $v = v_n = N^{-1/68}$.
Our goal is to show that

$$\sup_{x\in[a,b]} Nv_n|m_n - \mathsf{E}m_n| \to 0 \qquad a.s. \quad \text{as } n\to\infty \tag{4.1}$$

Write $D = B_n - zI$, $D_j = D - r_j r_j^*$, and $D_{j\underline{j}} = D - (r_j r_j^* + r_{\underline{j}} r_{\underline{j}}^*)$ $j \ne \underline{j}$. Then $m_n = \frac{1}{n}\mathsf{tr}\,(D^{-1})$. Let us also denote

$$\alpha_j = r_j^* D_j^{-2} r_j - N^{-1}\mathsf{tr}\,(D_j^{-2}T_n), \quad a_j = N^{-1}\mathsf{tr}\,(D_j^{-2}T_n),$$

$$\beta_j = \frac{1}{1 + r_j^* D_j^{-1} r_j}, \quad b_n = \frac{1}{1 + N^{-1}\mathsf{E}\mathsf{tr}\,(T_n D_1^{-1})},$$

$$\gamma_j = r_j^* D_j^{-1} r_j - N^{-1}\mathsf{E}(\mathsf{tr}\,(D_j^{-1}T_n)), \quad \hat{\gamma}_j = r_j^* D_j^{-1} r_j - N^{-1}\mathsf{tr}\,(D_j^{-1}T_n).$$

We first derive bounds on moments of γ_j and $\hat{\gamma}_j$. Using (3.2) we find for all $p \ge 2$

$$\mathsf{E}|\hat{\gamma}_j|^p \le K_p N^{-p}\mathsf{E}(\mathsf{tr}\,T_n^{1/2}D_j^{-1}T_n\bar{D}_j^{-1}T_n^{1/2})^{p/2} \le K_p N^{-p/2} v_n^{-p}. \tag{4.2}$$

Using Lemmas 2.2 and 2.10 we have for $p \ge 2$

$$\mathsf{E}|\gamma_j - \hat{\gamma}_j|^p = \mathsf{E}|\gamma_1 - \hat{\gamma}_1|^p = \mathsf{E}\left|\frac{1}{N}\sum_{j=2}^N \mathsf{E}_j\mathsf{tr}\,T_n D_1^{-1} - \mathsf{E}_{j-1}\mathsf{tr}\,T_n D_1^{-1}\right|^p$$

17

$$= \mathsf{E}\left|\frac{1}{N}\sum_{j=2}^{N}\mathsf{E}_j\operatorname{tr} T_n(D_1^{-1}-D_{1j}^{-1})-\mathsf{E}_{j-1}\operatorname{tr} T_n(D_1^{-1}-D_{1j}^{-1})\right|^p$$

$$= \mathsf{E}\left|\frac{1}{N}\sum_{j=2}^{N}(\mathsf{E}_j-\mathsf{E}_{j-1})\frac{r_j^*D_{1j}^{-1}T_nD_{1j}^{-1}r_j}{1+r_j^*D_{1j}^{-1}r_j}\right|^p \le K_p\frac{1}{N^p}\mathsf{E}\left(\sum_{j=2}^{N}\left|(\mathsf{E}_j-\mathsf{E}_{j-1})\frac{r_j^*D_{1j}^{-1}T_nD_{1j}^{-1}r_j}{1+r_j^*D_{1j}^{-1}r_j}\right|^2\right)^{p/2}$$

$$\le K_pN^{-p/2}v_n^{-p}.$$

Therefore

$$\mathsf{E}|\gamma_j|^p \le K_pN^{-p/2}v_n^{-p}. \tag{4.3}$$

We next prove that b_n is bounded for all n. We have b_n and β_1 both bounded in absolute value by $|z|/v_n$ (see (3.4)). From the equation relating $\underline{m}_n$ to the β_j's (above (3.4)) we have $\mathsf{E}\beta_1 = -z\mathsf{E}\underline{m}_n$. Using (3.24) we get

$$\sup_{x\in[a,b]}|\mathsf{E}(\underline{m}_n(z))-\underline{m}_n^0(z)| = o(v_n).$$

Since $\underline{m}_n^0$ is bounded for all n, $x \in [a,b]$ and v we have $\sup_{x\in[a,b]}|\mathsf{E}\beta_1| \le K$.

Since $b_n = \beta_1 + \beta_1 b_n\gamma_1$ we get

$$\sup_{x\in[a,b]}|b_n| = \sup_{x\in[a,b]}|\mathsf{E}\beta_1+\mathsf{E}\beta_1 b_n\gamma_1| \le K + K_2^{1/2}v_n^{-3}N^{-1/2} \le K.$$

Since $|m_n(x_1+iv_n)-m_n(x_2+iv_n)| \le |x_1-x_2|v_n^{-2}$, we see that (4.1) will follow from

$$\max_{x\in S_n}Nv_n|m_n - \mathsf{E}m_n| \to 0 \qquad a.s.$$

where S_n now contains n^2 elements, equally spaced in $[a,b]$.

We write

$$\mathsf{E}m_n - m_n = -\frac{1}{n}\sum_{j=1}^{N}\mathsf{E}_j\operatorname{tr} D^{-1} - \mathsf{E}_{j-1}\operatorname{tr} D^{-1}$$

$$= \frac{1}{n}\sum_{j=1}^{N}[\mathsf{E}_j - \mathsf{E}_{j-1}]\left(\frac{r_j^*D_j^{-2}r_j}{1+r_j^*D_j^{-1}r_j}\right)$$

$$= \frac{1}{n}\sum_{j=1}^{N}(\mathsf{E}_j - \mathsf{E}_{j-1})\frac{r_j^*D_j^{-2}r_j}{1+N^{-1}\mathsf{E}\operatorname{tr} T_nD_j^{-1}}$$

$$+\frac{1}{n}\sum_{j=1}^{N}(\mathsf{E}_j - \mathsf{E}_{j-1})\frac{r_j^*D_j^{-2}r_j(N^{-1}\mathsf{E}\operatorname{tr} T_nD_j^{-1}-r_j^*D_j^{-1}r_j)}{(1+N^{-1}\mathsf{E}\operatorname{tr} T_nD_j^{-1})^2}$$

18

$$+\frac{1}{n}\sum_{j=1}^{N}(\mathsf{E}_j-\mathsf{E}_{j-1})\frac{r_j^*D_j^{-2}r_j(N^{-1}\mathsf{E}\operatorname{tr}T_nD_j^{-1}-r_j^*D_j^{-1}r_j)^2}{(1+N^{-1}\mathsf{E}\operatorname{tr}T_nD_j^{-1})^2(1+r_j^*D_j^{-1}r_j)}$$

$$=\frac{b_n}{n}\sum_{j=1}^{N}\mathsf{E}_j\alpha_j-\frac{b_n^2}{n}\sum_{j=1}^{N}\mathsf{E}_ja_j\hat{\gamma}_j$$

$$-\frac{b_n^2}{n}\sum_{j=1}^{N}(\mathsf{E}_j-\mathsf{E}_{j-1})(\alpha_j\gamma_j-r_j^*D_j^{-2}r_j\beta_j\gamma_j^2)$$

$$\equiv W_1-W_2-W_3.$$

Let F_{nj} be the spectral distribution of the matrix $\sum_{k\neq j}r_kr_k^*$. From Lemma 2.12 and (3.27) we get

$$\max_j\mathsf{E}_j(F_{nj}([a',b']))^2=o(N^{-2/17})=o(v_n^8)\quad a.s. \tag{4.4}$$

Define

$$\mathcal{B}_j=I_{[\mathsf{E}_{j-1}F_{nj}([a',b'])\leq v_n^4]\cap[\mathsf{E}_{j-1}(F_{nj}([a',b']))^2\leq v_n^8]}.$$

Then $\mathcal{B}_j=I_{[\mathsf{E}_jF_{nj}([a',b'])\leq v_n^4]\cap[\mathsf{E}_j(F_{nj}([a',b']))^2\leq v_n^8]}$ $a.s.$ and we have

$$\mathsf{P}\left(\bigcup_{j=1}^{N}[\mathcal{B}_j=0],i.o.\right)=0.$$

Therefore we have for any $\varepsilon>0$,

$$P\left(\max_{x\in S_n}|Nv_nW_1|>\varepsilon,i.o.\right)$$

$$\leq P\left(\left(\left[\max_{x\in S_n}\left|v_n\sum_{j=1}^{N}\mathsf{E}_j(\alpha_j)\right|>\underline{\varepsilon}\right]\bigcap_{j=1}^{N}[\mathcal{B}_j=1]\right)\bigcup\left(\bigcup_{j=1}^{N}[\mathcal{B}_j=0]\right),i.o.\right)$$

$$\leq\mathsf{P}\left(\max_{x\in S_n}\left|v_n\sum_{j=1}^{N}\mathsf{E}_j(\alpha_j)\mathcal{B}_j\right|>\underline{\varepsilon},i.o.\right),$$

where $\underline{\varepsilon}=\inf_n n\varepsilon/(Nb_n)>0$ since b_n is bounded. Note that for each $x\in\mathbb{R}$ $\{\mathsf{E}_j(\alpha_j)\mathcal{B}_j\}$ forms a martingale difference sequence.

By Lemma 2.1 and (3.2), we have for each $x\in[a,b]$ and $p\geq 2$

$$\mathsf{E}\left|v_n\sum_{j=1}^{N}\mathsf{E}_j(\alpha_j)\mathcal{B}_j\right|^p\leq K_p\left(\mathsf{E}\left(\sum_{j=1}^{N}\mathsf{E}_{j-1}|v_n\mathsf{E}_j(\alpha_j)\mathcal{B}_j|^2\right)^{p/2}+\sum_{j=1}^{N}\mathsf{E}|v_n\mathsf{E}_j(\alpha_j)\mathcal{B}_j|^p\right)$$

$$\leq K_p \left(\mathsf{E} \left(\sum_{j=1}^{N} \mathsf{E}_{j-1} v_n^2 N^{-2} \mathcal{B}_j \mathrm{tr}\, (T_n^{1/2} D_j^{-2} T_n \overline{D}_j^{-2} T_n^{1/2}) \right)^{p/2} + N v_n^p \mathsf{E}|\alpha_1|^p \right)$$

$$\leq K_p \left(v_n^p N^{-p} \mathsf{E} \left(\sum_{j=1}^{N} \mathcal{B}_j \mathsf{E}_{j-1} \mathrm{tr}\, (D_j^{-2} \overline{D}_j^{-2}) \right)^{p/2} + N v_n^p N^{-p} \mathsf{E}(\mathrm{tr}\, (T_n^{1/2} D_1^{-2} T_n \overline{D}_1^{-2} T_n^{1/2}))^{p/2} \right)$$

$$\leq K_p \left(v_n^p N^{-p} \mathsf{E} \left(\sum_{j=1}^{N} \mathcal{B}_j \mathsf{E}_{j-1} \mathrm{tr}\, (D_j^{-2} \overline{D}_j^{-2}) \right)^{p/2} + v_n^{-p} N^{1-p/2} \right).$$

Let λ_{kj} denote the kth smallest eigenvalue of $\sum_{k \neq j} r_k r_k^*$. We have

$$\sum_{j=1}^{N} \mathcal{B}_j \mathsf{E}_{j-1} \mathrm{tr}\, D_j^{-2} \overline{D}_j^{-2}$$

$$= \sum_{j=1}^{N} \mathcal{B}_j \mathsf{E}_{j-1} \left[\sum_{\lambda_{kj} \notin [a'b']} \frac{1}{((\lambda_{kj} - x)^2 + v_n^2)^2} + \sum_{\lambda_{kj} \in [a'b']} \frac{1}{((\lambda_{kj} - x)^2 + v_n^2)^2} \right]$$

$$\leq \sum_{j=1}^{N} (n \underline{\epsilon}^{-4} + \mathcal{B}_j v_n^{-4} \mathsf{E}_{j-1} n F_{nj}([a', b'])) \leq K N^2.$$

Therefore

$$\mathsf{P} \left(\max_{x \in S_n} \left| v_n \sum_{j=1}^{N} \mathsf{E}_j(\alpha_j) \mathcal{B}_j \right| > \underline{\varepsilon} \right) \leq n^2 K_{p,\underline{\varepsilon}} N^{-p/68}$$

which is summable when $p > 204$. Therefore, $\max_{x \in S_n} |W_1| = o(1/Nv_n)$ a.s.

Proving

$$\max_{x \in S_n} |W_2| = o(1/Nv_n) \quad a.s. \tag{4.5}$$

is handled the same way. We get using Lemma 2.1, (3.2), and the fact that $|a_j| \leq \frac{n}{N} v_n^{-2}$

$$\mathsf{E} \left| v_n \sum_{j=1}^{N} \mathsf{E}_j(a_j \hat{\gamma}_j) \mathcal{B}_j \right|^p \leq K_p \left(\mathsf{E} \left(\sum_{j=1}^{N} \mathsf{E}_{j-1} |v_n \mathsf{E}_j(a_j \hat{\gamma}_j) \mathcal{B}_j|^2 \right)^{p/2} + \sum_{j=1}^{N} \mathsf{E}|v_n \mathsf{E}_j(a_j \hat{\gamma}_j) \mathcal{B}_j|^p \right)$$

$$\leq K_p \left(v_n^p N^{-p} \mathsf{E} \left(\sum_{j=1}^{N} \mathcal{B}_j \mathsf{E}_{j-1} (|a_j|^2 \mathrm{tr}\, D_j^{-1} \overline{D}_j^{-1}) \right)^{p/2} + N^{1-p} v_n^{-p} (\mathrm{tr}\, D_j^{-1} \overline{D}_j^{-1})^{p/2} \right)$$

$$\leq K_p \left(v_n^p N^{-p} \mathsf{E} \left(\sum_{j=1}^{N} \mathcal{B}_j \mathsf{E}_{j-1} (|a_j|^2 \mathrm{tr}\, D_j^{-1} \overline{D}_j^{-1}) \right)^{p/2} + v_n^{-2p} N^{1-p/2} \right).$$

20

This time

$$\sum_{j=1}^{N} \mathcal{B}_j \mathsf{E}_{j-1}(|a_j|^2 \operatorname{tr} D_j^{-1}\overline{D}_j^{-1}) \leq \sum_{j=1}^{N} \mathcal{B}_j \mathsf{E}_{j-1} N^{-2} n \sum_{k} \frac{1}{((\lambda_{kj}-x)^2 + v_n^2)^2} \sum_{k} \frac{1}{(\lambda_{kj}-x)^2 + v_n^2}$$

$$\leq \sum_{j=1}^{N} \mathcal{B}_j N^{-2} n \mathsf{E}_{j-1}(n\underline{\epsilon}^{-4} + v_n^{-4} n F_{nj}([a,b]))(n\underline{\epsilon}^{-2} + v_n^{-2} n F_{nj}([a,b])) \leq K N^2,$$

so that (4.5) also holds.

Using Lemmas 2.2, 2.10, (3.2), and (4.3) we get

$$\mathsf{E}\left|v_n \sum_{j=1}^{N}(\mathsf{E}_j - \mathsf{E}_{j-1})(\alpha_j\gamma_j - r_j D_j^{-2} r_j \beta_j \gamma_j^2)\right|^p \leq K_p v_n^p N^{p/2}(\mathsf{E}|\alpha_1\gamma_1|^p + v_n^{-p}\mathsf{E}|\gamma_1|^{2p})$$

$$\leq K_p v_n^p N^{p/2}(N^{-p}(\mathsf{E}(\operatorname{tr} D_1^{-2}\overline{D}_1^{-2})^p)^{1/2} N^{-p/2} v_n^{-p} + v_n^{-3p} N^{-p})$$

$$\leq K_p v_n^p N^{p/2}(N^{-p} N^{p/2} v_n^{-2p} N^{-p/2} v_n^{-p} + v_n^{-3p} N^{-p}) = 2K_p N^{-p/2} v_n^{-2p}.$$

Thus we get $\max_{x \in S_n} |W_3| = o(1/Nv_n)$ $a.s.$ and, consequently, (4.1).

5. Convergence of expected value

Our next goal is to show that, for $v = N^{-1/68}$.

$$\sup_{x \in [a,b]} |\mathsf{E}\underline{m}_n - \underline{m}_n^0| = O(1/N). \tag{5.1}$$

We begin by deriving an identity similar to (3.5). Write $B_n - zI - (-z\mathsf{E}\underline{m}_n(z)T_n - zI)$ $= \sum_{j=1}^N r_j r_j^* - (-z\mathsf{E}\underline{m}_n(z))T_n$. Taking first inverses and then expected value we get

$$(-z\mathsf{E}\underline{m}_n T_n - zI)^{-1} - \mathsf{E}(B_n - zI)^{-1}$$

$$= (-z\mathsf{E}\underline{m}_n T_n - zI)^{-1}\mathsf{E}\left[\sum_{j=1}^N r_j r_j^* - (-z\mathsf{E}\underline{m}_n(z))T_n(B_n - zI)^{-1}\right]$$

$$= -z^{-1} \sum_{j=1}^N \mathsf{E}\beta_j \left[(\mathsf{E}\underline{m}_n(z)T_n + I)^{-1} r_j r_j^* (B_{(j)} - zI)^{-1} - \frac{1}{N}(\mathsf{E}\underline{m}_n(z)T_n + I)^{-1}T_n\mathsf{E}(B_n - zI)^{-1}\right]$$

$$= -z^{-1} N\mathsf{E}\beta_1 \left[(\mathsf{E}\underline{m}_n(z)T_n + I)^{-1} r_1 r_1^* D_1^{-1} - \frac{1}{N}(\mathsf{E}\underline{m}_n(z)T_n + I)^{-1}T_n\mathsf{E}D^{-1}\right].$$

Taking the trace on both sides and dividing by $-N/z$, we get

$$c_n \int \frac{dH_n(t)}{1 + t\mathsf{E}\underline{m}_n} + zc_n\mathsf{E}(m_n(z))$$

$$= \mathsf{E}\beta_1 \left[r_1^* D_1^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1} r_1 - \frac{1}{N}\mathsf{E}\mathsf{tr}\,(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D^{-1}\right]. \tag{5.2}$$

We first show

$$\sup_{x \in [a,b]} N^{-1}\left|\mathsf{E}\mathsf{tr}\,(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D^{-1} - \mathsf{E}\mathsf{tr}\,(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_1^{-1}\right| = O(N^{-1}). \tag{5.3}$$

From (4.4) we get

$$\sup_{x \in [a,b]} \mathsf{E}(\mathsf{tr}\, D_1^{-1}\overline{D}_1^{-1})^2 \leq \mathsf{E}(n\underline{\epsilon}^{-2} + v_n^{-2}nF_{n1}([a',b']))^2 \leq KN^2 \tag{5.4}$$

and

$$\sup_{x \in [a,b]} \mathsf{E}\mathsf{tr}\, D_1^{-2}\overline{D}_1^{-2} \leq \mathsf{E}(n\underline{\epsilon}^{-4} + v_n^{-4}nF_{n1}([a',b'])) \leq KN. \tag{5.5}$$

Also, because of (3.24) and the fact that $-1/\underline{m}_n^0(z)$ stays uniformly away from the eigenvalues of T_n for all $x \in [a,b]$, we must have

$$\sup_{x \in [a,b]} \|(\mathsf{E}\underline{m}_n(z)T_n + I)^{-1}\| \leq K \tag{5.6}$$

22

Therefore, from (3.3), (4.3), (5.4)–(5.6), and the fact that $\sup_{x\in[a,b]}|b_n|$ is bounded, we get

$$\text{left side of (5.3)} = N^{-1}\sup_{x\in[a,b]}|\mathsf{E}\beta_1 r_1^* D_1^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_1^{-1}r_1|$$

$$\leq N^{-1}\sup_{x\in[a,b]}(|b_n|\cdot|\mathsf{E}r_1^* D_1^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_1^{-1}r_1|$$

$$+\mathsf{E}|\beta_1 b_n\gamma_1 r_1^* D_1^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_1^{-1}r_1|)$$

$$\leq KN^{-1}\sup_{x\in[a,b]}(N^{-1}|\mathsf{E}\mathrm{tr}\,T_n^{1/2}D_1^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_1^{-1}T_n^{1/2}|$$

$$+v_n^{-1}(\mathsf{E}|\gamma_1|^2)^{1/2}(E|r_1^* D_1^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_1^{-1}r_1|^2)^{1/2})$$

$$\leq KN^{-1}\sup_{x\in[a,b]}(N^{-1}\mathsf{E}\mathrm{tr}\,D_1^{-1}\overline{D}_1^{-1} + v_n^{-1}N^{-1/2}v_n^{-1}N^{-1}(\mathrm{Etr}\,D_1^{-2}\overline{D}_1^{-2} + \mathsf{E}(\mathrm{tr}\,D_1^{-1}\overline{D}_1^{-1})^2)^{1/2})$$

$$\leq KN^{-1}.$$

Thus (5.3) holds.

From (3.2), (5.4), and (5.6) we get

$$\sup_{x\in[a,b]}\mathsf{E}|r_1^* D_1^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}r_1 - N^{-1}\mathrm{tr}\,D_1^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n|^2$$

$$\leq KN^{-2}\sup_{x\in[a,b]}\mathrm{Etr}\,D_1^{-1}\overline{D}_1^{-1} \leq KN^{-1} \quad (5.7)$$

Next we show

$$\sup_{x\in[a,b]}N^{-2}\mathsf{E}|\mathrm{tr}\,(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_1^{-1} - \mathrm{Etr}\,(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_1^{-1}|^2 \leq KN^{-1}. \quad (5.8)$$

Let

$$\beta_{1j} = \frac{1}{1 + r_j^* D_{1j}^{-1}r_j}, \quad b_{1n} = \frac{1}{1 + N^{-1}\mathrm{Etr}\,(T_n D_{12}^{-1})}, \quad \text{and } \gamma_{1j} = r_j^* D_{1j}^{-1}r_j - N^{-1}\mathsf{E}(\mathrm{tr}\,(D_{1j}^{-1}T_n)).$$

As in the previous section, both β_{1j} and b_{1n} are bounded in absolute value by $|z|/v_n$ and γ_{1j} satisfies the same bound as in (4.3). Moreover, if we let $X_{(1)}$ denote X without its first column, then one can easily derive

$$\frac{1}{N-1}\mathrm{tr}\,\left(\frac{1}{N-1}X_{(1)}^* T_n X_{(1)} - \left(\frac{N}{N-1}z\right)I\right)^{-1} = \frac{1}{N}\mathrm{tr}\,\left(\frac{1}{N}X_{(1)}^* T_n X_{(1)} - zI\right)^{-1} = -\frac{1}{z(N-1)}\sum_{j=2}^{N}\beta_{1j},$$

and conclude that $\sup_{x\in[a,b]}|\mathsf{E}\beta_{1j}|$ and consequently $\sup_{x\in[a,b]}|b_{1n}|$ are bounded.

It is also clear that the bounds in (4.4), (5.4), and (5.5) hold when two columns of X are removed. Moreover, with F_{n12} denoting the e.d.f of $\sum_{j\neq 1,2} r_j r_j^*$ we get

$$\sup_{x\in[a,b]} \mathsf{E}(\operatorname{tr} D_{12}^{-1}\overline{D}_{12}^{-1})^4 \leq \mathsf{E}(n\underline{\epsilon}^{-2} + v_n^{-2}nF_{n12}([a',b']))^4$$

$$\leq KN^4(\underline{\epsilon}^{-8} + v_n^{-8}\mathsf{E}(F_{n12}([a',b']))^2) \leq KN^4$$

and

$$\sup_{x\in[a,b]} \mathsf{E}(\operatorname{tr} D_{12}^{-2}\overline{D}_{12}^{-2})^2 \leq \mathsf{E}(n\underline{\epsilon}^{-4} + v_n^{-4}nF_{n12}([a',b']))^2 \leq KN^2.$$

With these facts and (3.3) and (5.6) we have

$$\text{left side of (5.8)} \; = \; \sup_{x\in[a,b]} N^{-2}\sum_{j=2}^{N} \mathsf{E}|(\mathsf{E}_j - \mathsf{E}_{j-1})\operatorname{tr}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_1^{-1}|^2$$

$$\leq 2N^{-2}\sup_{x\in[a,b]}\sum_{j=2}^{N} \mathsf{E}|\beta_{1j}r_j^* D_{1j}^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_{1j}^{-1}r_j|^2$$

$$= 2N^{-1}\sup_{x\in[a,b]} \mathsf{E}|(b_{1n} - \beta_{12}b_{1n}\gamma_{12})r_2^* D_{12}^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_{12}^{-1}r_2|^2$$

$$\leq KN^{-1}\Big(\sup_{x\in[a,b]} \mathsf{E}|r_2^* D_{12}^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_{12}^{-1}r_2|^2$$

$$+ v_n^{-2}(\mathsf{E}|\gamma_{12}|^4 \mathsf{E}|r_2^* D_{12}^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1}T_n D_{12}^{-1}r_2|^4)^{1/2}\Big)$$

$$\leq KN^{-3}\sup_{x\in[a,b]} \Big(\mathsf{E}(\operatorname{tr} D_{12}^{-2}\overline{D}_{12}^{-2})$$

$$+ \mathsf{E}(\operatorname{tr} D_{12}^{-1}\overline{D}_{12}^{-1})^2 + v_n^{-2}N^{-1}v_n^{-2}(\mathsf{E}\operatorname{tr}(D_{12}^{-2}\overline{D}_{12}^{-2})^2 + \mathsf{E}(\operatorname{tr} D_{12}^{-1}\overline{D}_{12}^{-1})^4)^{1/2}\Big)$$

$$\leq KN^{-3}(N^2 + Nv_n^{-4}) \leq KN^{-1}.$$

Thus we get (5.8)

Notice we get the same result if $(\mathsf{E}\underline{m}_n T_n + I)^{-1}$ is removed from all the expressions, that is, we have just shown

$$\sup_{x\in[a,b]} \mathsf{E}|\gamma_1 - \hat{\gamma}_1|^2 \leq KN^{-1}.$$

Moreover, from (3.2) and (5.4), when $p = 2$

$$\sup_{x\in[a,b]} \mathsf{E}|\hat{\gamma}_1|^2 \leq \sup_{x\in[a,b]} KN^{-2}\mathsf{E}\operatorname{tr} D_1^{-1}\overline{D}_1^{-1} \leq KN^{-1}.$$

24

Therefore
$$\sup_{x \in [a,b]} \mathsf{E}|\gamma_1|^2 \le KN^{-1}. \tag{5.9}$$

From (4.3), (5.2), (5.3), (5.7)–(5.9) we get

$$\sup_{x \in [a,b]} \left| c_n \int \frac{dH_n(t)}{1 + t\mathsf{E}\underline{m}_n} + zc_n\mathsf{E}(m_n(z)) \right|$$

$$\le KN^{-1} + \sup_{x \in [a,b]} \left| \mathsf{E}\beta_1 \left[r_1^* D_1^{-1} (\mathsf{E}\underline{m}_n T_n + I)^{-1} r_1 - \frac{1}{N}\mathsf{Etr}\,(\mathsf{E}\underline{m}_n T_n + I)^{-1} T_n D_1^{-1} \right] \right|$$

$$= KN^{-1} + \sup_{x \in [a,b]} |b_n|^2 \left| \mathsf{E}(\gamma_1 - \beta_1 \gamma_1^2) \left[r_1^* D_1^{-1}(\mathsf{E}\underline{m}_n T_n + I)^{-1} r_1 - \frac{1}{N}\mathsf{Etr}\,(\mathsf{E}\underline{m}_n T_n + I)^{-1} T_n D_1^{-1} \right] \right|$$

$$\le K(N^{-1} + \sup_{x \in [a,b]} (\mathsf{E}|\gamma_1|^2 + v_n^{-2}\mathsf{E}|\gamma_1|^4)^{1/2} N^{-1/2})$$

$$\le K(N^{-1} + (N^{-1} + v_n^{-2} N^{-2} v_n^{-4})^{1/2} N^{-1/2}) \le KN^{-1}.$$

As in Section 3 we let

$$w_n = -\frac{1}{z} \int \frac{dH_n(t)}{1 + t\mathsf{E}\underline{m}_n(z)} - \mathsf{E}(m_n(z))$$

and

$$\omega_n = -z - \frac{1}{\mathsf{E}\underline{m}_n} + c_n \int \frac{t\,dH_n(t)}{1 + t\mathsf{E}\underline{m}_n}.$$

Then

$$\sup_{x \in [a,b]} |w_n| \le KN^{-1},$$

$\omega_n = w_n z c_n / \mathsf{E}\underline{m}_n$, and equation (3.20) together with the steps leading to (3.21) hold with $\underline{m}_n$ replaced with its expected value. From (3.10) it is clear that $\underline{m}_n^0$ must be uniformly bounded away from 0 for all $x \in [a,b]$ and all n. From (3.24) we see that $\mathsf{E}\underline{m}_n$ must also satisfy this same property. Therefore

$$\sup_{x \in [a,b]} |\omega_n| \le KN^{-1}.$$

Using (3.11), (3.12), (3.14), and (3.15) it follows that $\sup_{x \in [a,b]} |\underline{m}_n^0|$ is bounded in n and

$$\sup_{x \in [a,b]} \frac{\underline{m}_2^0 c_n \int \frac{t^2\,dH_n(t)}{|1 + t\underline{m}_n^0|^2}}{v_n + \underline{m}_2^0 c_n \int \frac{t^2\,dH_n(t)}{|1 + t\underline{m}_n^0|^2}}$$

25

is bounded away from 1 for all n. Therefore, we get for all n sufficiently large

$$\sup_{x\in[a,b]} |\mathsf{E}\underline{m}_n - \underline{m}_n^0| \le Kc_n z\underline{m}_n^0 w_n \le KN^{-1},$$

which is (5.1).

6. Completing the proof

From the last two sections we get

$$\sup_{x\in[a,b]} |\underline{m}_n(z) - \underline{m}_n^0(z)| = o(1/Nv_n) \quad a.s. \tag{6.1}$$

when $v_n = N^{-1/68}$. It is clear from the arguments used in Sections 3–5 that (6.1) is true when the imaginary part of z is replaced by a constant multiple of v_n. In fact we have

$$\max_{k\in\{1,2\dots,34\}} \sup_{x\in[a,b]} |\underline{m}_n(x+i\sqrt{k}v_n) - \underline{m}_n^0(x+i\sqrt{k}v_n)| = o(1/Nv_n) = o(v_n^{67}) \quad a.s.$$

We take the imaginary part and get

$$\max_{k\in\{1,2\dots,34\}} \sup_{x\in[a,b]} \left| \int \frac{d(F^{\underline{B}_n}(\lambda) - F^{c_n,H_n}(\lambda))}{(x-\lambda)^2 + kv_n^2} \right| = o(v_n^{66}) \quad a.s.$$

Upon taking differences we find

$$\max_{k_1\ne k_2} \sup_{x\in[a,b]} \left| \int \frac{v_n^2\, d(F^{\underline{B}_n}(\lambda) - F^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + k_1 v_n^2)((x-\lambda)^2 + k_2 v_n^2)} \right| = o(v_n^{66}) \quad a.s.$$

$$\max_{\substack{k_1,k_2,k_3\\ \text{distinct}}} \sup_{x\in[a,b]} \left| \int \frac{(v_n^2)^2\, d(F^{\underline{B}_n}(\lambda) - F^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + k_1 v_n^2)((x-\lambda)^2 + k_2 v_n^2)((x-\lambda)^2 + k_3 v_n^2)} \right| = o(v_n^{66}) \quad a.s.$$

$$\vdots$$

$$\sup_{x\in[a,b]} \left| \int \frac{(v_n^2)^{33}\, d(F^{\underline{B}_n}(\lambda) - F^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + v_n^2)((x-\lambda)^2 + 2v_n^2)\cdots((x-\lambda)^2 + 34v_n^2)} \right| = o(v_n^{66}) \quad a.s.$$

Thus

$$\sup_{x\in[a,b]} \left| \int \frac{d(F^{\underline{B}_n}(\lambda) - F^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + v_n^2)((x-\lambda)^2 + 2v_n^2)\cdots((x-\lambda)^2 + 34v_n^2)} \right| = o(1) \quad a.s.$$

26

We split up the integral and get

$$
\sup_{x\in[a,b]}\left|\int \frac{I_{[a',b']^c}\,d(F^{\underline{B}_n}(\lambda) - F^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + v_n^2)((x-\lambda)^2 + 2v_n^2)\cdots((x-\lambda)^2 + 34v_n^2)}\right.
$$

$$
\left. + \sum_{\lambda_j\in[a',b']} \frac{v_n^{68}}{((x-\lambda_j)^2 + v_n^2)((x-\lambda_j)^2 + 2v_n^2)\cdots((x-\lambda_j)^2 + 34v_n^2)}\right| = o(1) \quad a.s. \quad (6.2)
$$

Now if, for each term in a subsequence satisfying (6.2), there is at least one eigenvalue contained in $[a,b]$, then the sum in (6.2), with x evaluated at these eigenvalues, will be uniformly bounded away from 0. Thus, at these same x values, the integral in (6.2) must also stay uniformly bounded away from 0. But the integral converges to zero $a.s.$ since the integrand is bounded and with probability one, both $F^{\underline{B}_n}$ and F^{c_n,H_n} converge weakly to the same limit having no mass on $\{a',b'\}$. Thus, with probability one, no eigenvalues of B_n will appear in $[a,b]$ for all n sufficiently large. This completes the proof of Theorem 1.1.

APPENDIX

We give here a proof Lemma 2.7. We first prove
Lemma A.1 For $X = (X_1,\ldots,X_n)^T$ i.i.d. standardized (complex) entries, B $n\times n$ Hermitian non-negative definite matrix we have for any $p \geq 1$

$$
\mathsf{E}|X^*BX|^p \leq K_p\left((\mathsf{tr}\,B)^p + \mathsf{E}|X_1|^{2p}\mathsf{tr}\,B^p\right). \tag{A.1}
$$

Proof. Notice the result is trivially true for $p = 1$. For $p > 1$ we have

$$
\mathsf{E}|X^*BX|^p \leq K_p\left(\mathsf{E}\left|\sum_{i=1}^n |X_i|^2 B_{ii}\right|^p + \mathsf{E}\left|\sum_{i=2}^n \overline{X}_i \sum_{j<i} X_j B_{ij}\right|^p + \mathsf{E}\left|\sum_{j=2}^n X_j \sum_{i<j} \overline{X}_i B_{ij}\right|^p\right)
$$

$$
= K_p\left(\mathsf{E}\left|\sum_{i=1}^n |X_i|^2 B_{ii}\right|^p + 2\mathsf{E}\left|\sum_{i=2}^n \overline{X}_i \sum_{j<i} X_j B_{ij}\right|^p\right). \tag{A.2}
$$

Using Lemmas 2.3 and 2.9

$$
\mathsf{E}\left|\sum_{i=1}^n |X_i|^2 B_{ii}\right|^p \leq K_p\left((\mathsf{tr}\,B)^p + \sum_{i=1}^n \mathsf{E}|X_1|^{2p}(B_{ii})^p\right)
$$

$$
\leq K_p\left((\mathsf{tr}\,B)^p + \mathsf{E}|X_1|^{2p}\mathsf{tr}\,B^p\right).
$$

27

For $1 < p \leq 2$ we have using Lemma 2.2

$$\mathsf{E}\left|\sum_{i=2}^{n}\overline{X}_i\sum_{j<i}X_jB_{ij}\right|^p \leq K_p\mathsf{E}\left(\sum_{i=2}^{n}\left|\overline{X}_i\sum_{j<i}X_jB_{ij}\right|^2\right)^{p/2} \leq K_p\left(\sum_{i=2}^{n}\mathsf{E}\left|\overline{X}_i\sum_{j<i}X_jB_{ij}\right|^2\right)^{p/2}$$

$$= K_p\left(\sum_{j<i}|B_{ij}|^2\right)^{p/2} \leq K_p(\operatorname{tr} B^2)^{p/2} \leq K_p(\operatorname{tr} B)^p.$$

Therefore (A.1) is true for $p \in [1,2]$. We proceed by induction on k, where $p \in [2^k, 2^{k+1}]$. Assume (A.1) is true for $p \in [2^{k-1}, 2^k]$, and suppose $p \in [2^k, 2^{k+1}]$. Since the first term in (A.2) is bounded by the right side of (A.1), we need only consider the second term. Let $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$. We have by Lemma 2.1

$$\mathsf{E}\left|\sum_{i=2}^{n}\overline{X}_i\sum_{j<i}X_jB_{ij}\right|^p \leq K_p\left(\mathsf{E}\left(\sum_{i=2}^{n}\left|\sum_{j<i}X_jB_{ij}\right|^2\right)^{p/2} + \mathsf{E}|X_1|^p\sum_{i=2}^{n}\mathsf{E}\left|\sum_{j<i}X_jB_{ij}\right|^p\right).$$

Using Lemma 2.4 (with $q = 2$) we have

$$\mathsf{E}\left(\sum_{i=2}^{n}\left|\sum_{j<i}X_jB_{ij}\right|^2\right)^{p/2} = \mathsf{E}\left(\sum_{i=2}^{n}\left|\mathsf{E}\left(\sum_{j=1}^{n}X_jB_{ij}\middle|\mathcal{F}_{i-1}\right)\right|^2\right)^{p/2}$$

$$\leq K_p\mathsf{E}\left(\sum_{i=2}^{n}\left|\sum_{j=1}^{n}X_jB_{ij}\right|^2\right)^{p/2} \leq K_p\mathsf{E}\left(X^*B^2X\right)^{p/2}$$

$$\leq \text{ (by the inductive hyp.) } K_p\left((\operatorname{tr} B^2)^{p/2} + \mathsf{E}|X_1|^{2(p/2)}\operatorname{tr} B^{2(p/2)}\right)$$

$$\leq K_p\left((\operatorname{tr} B)^p + \mathsf{E}|X_1|^{2p}\operatorname{tr} B^p\right)$$

(using $1 \leq \mathsf{E}|X_1|^s \leq (\mathsf{E}|X_1|^{2s})^{1/2} \leq \mathsf{E}|X_1|^{2s}$ for $s \geq 2$).

Using Lemma 2.1 we have

$$\mathsf{E}|X_1|^p\sum_{i=2}^{n}\mathsf{E}\left|\sum_{j<i}X_jB_{ij}\right|^p \leq K_p\mathsf{E}|X_1|^p\sum_{i=2}^{n}\left(\left(\sum_{j<i}|B_{ij}|^2\right)^{p/2} + \mathsf{E}|X_1|^p\sum_{j<i}|B_{ij}|^p\right)$$

$$\leq K_p\mathsf{E}|X_1|^p(1 + \mathsf{E}|X_1|^p)\sum_{i=2}^{n}\left(\sum_{j<i}|B_{ij}|^2\right)^{p/2} \leq K_p\mathsf{E}|X_1|^p(1 + \mathsf{E}|X_1|^p)\sum_{i=1}^{n}((B^2)_{ii})^{p/2}$$

$$\leq \text{ (by Lemma 2.9) } \leq K_p\mathsf{E}|X_1|^p(1 + \mathsf{E}|X_1|^p)\operatorname{tr} B^p \leq K_p\mathsf{E}|X_1|^{2p}\operatorname{tr} B^p.$$

Therefore (A.1) is true for $p \in [2^k, 2^{k+1}]$ and the proof of the lemma is complete.

28

We can now prove Lemma 2.7 We have

$$\mathsf{E}|X^*CX - \operatorname{tr} C|^p$$

$$\leq K_p\left(\mathsf{E}\left|\sum_{i=1}^{n}\left(|X_i|^2 - 1\right)C_{i\,i}\right|^p + \mathsf{E}\left|\sum_{i=2}^{n}\overline{X}_i\sum_{j<i}X_j C_{i\,j}\right|^p + \mathsf{E}\left|\sum_{j=2}^{n}X_j\sum_{i<j}\overline{X}_i C_{i\,j}\right|^p\right).$$

Using Lemma 2.1

$$\mathsf{E}\left|\sum_{i=1}^{n}\left(|X_i|^2 - 1\right)C_{i\,i}\right|^p \leq K_p\left(\left(\sum_{i=1}^{n}\mathsf{E}\left(|X_i|^2 - 1\right)^2|C_{i\,i}|^2\right)^{p/2} + \sum_{i=1}^{n}\mathsf{E}\left||X_1|^2 - 1\right|^p|C_{i\,i}|^p\right)$$

$$\leq K_p\left(\left(\mathsf{E}|X_1|^4\operatorname{tr} CC^*\right)^{p/2} + \mathsf{E}|X_1|^{2p}\sum_{i=1}^{n}|C_{i\,i}|^p\right).$$

(using $\left(\mathsf{E}\left||X_1|^2 - 1\right|^p\right)^{1/p} \leq \left(\mathsf{E}|X_1|^{2p}\right)^{1/p} + 1 \leq 2\left(\mathsf{E}|X_1|^{2p}\right)^{1/p}$). From Lemma 2.9 we have

$$\sum_{i=1}^{n}|C_{i\,i}|^p \leq \sum_{i=1}^{n}(CC^*)_{i\,i}^{p/2} \leq \sum_{i=1}^{n}\lambda_i(CC^*)^{p/2}.$$

Therefore

$$\mathsf{E}\left|\sum_{i=1}^{n}\left(|X_i|^2 - 1\right)C_{i\,i}\right|^p \leq K_p\left(\left(\mathsf{E}|X_1|^4\operatorname{tr} CC^*\right)^{p/2} + \mathsf{E}|X_1|^{2p}\operatorname{tr}(CC^*)^{p/2}\right).$$

Using Lemma 2.1

$$\mathsf{E}\left|\sum_{i=2}^{n}\overline{X}_i\sum_{j<i}X_j C_{i\,j}\right|^p \leq K_p\left(\mathsf{E}\left(\sum_{i=2}^{n}\left|\sum_{j<i}X_j C_{i\,j}\right|^2\right)^{p/2} + \mathsf{E}|X_1|^p\sum_{i=2}^{n}\mathsf{E}\left|\sum_{j<i}X_j C_{i\,j}\right|^p\right).$$

Using Lemma 2.4 (with $q = 2$),

$$\mathsf{E}\left(\sum_{i=2}^{n}\left|\sum_{j<i}X_j C_{i\,j}\right|^2\right)^{p/2} = \mathsf{E}\left(\sum_{i=2}^{n}\left|\mathsf{E}\left(\sum_{j=1}^{n}X_j C_{i\,j}\Big|\mathcal{F}_{i-1}\right)\right|^2\right)^{p/2}$$

$$\leq K_p\mathsf{E}\left(\sum_{i=2}^{n}\left|\sum_{j=1}^{n}X_j C_{i\,j}\right|^2\right)^{p/2} \leq K_p\mathsf{E}(X^*C^*CX)^{p/2}$$

$$\leq \ (\text{by lemma A.1})\ K_p\left((\operatorname{tr} C^*C)^{p/2} + \mathsf{E}|X_1|^{2p}\operatorname{tr}(C^*C)^{p/2}\right)$$

$$= K_p\left((\operatorname{tr} CC^*)^{p/2} + \mathsf{E}|X_1|^{2p}\operatorname{tr}(CC^*)^{p/2}\right).$$

29

Using Lemma 2.1 we have

$$\mathsf{E}|X_1|^p \sum_{i=2}^{n} \mathsf{E}\left|\sum_{j<i} X_j C_{ij}\right|^p \le K_p \mathsf{E}|X_1|^p \sum_{i=2}^{n}\left(\left(\sum_{j<i}|C_{ij}|^2\right)^{p/2} + \mathsf{E}|X_1|^p \sum_{j<i}|C_{ij}|^p\right)$$

$$\le K_p \mathsf{E}|X_1|^p(1+\mathsf{E}|X_1|^p)\sum_{i=2}^{n}\left(\sum_{j<i}|C_{ij}|^2\right)^{p/2} \le K_p \mathsf{E}|X_1|^p(1+\mathsf{E}|X_1|^p)\sum_{i=1}^{n}((CC^*)_{ii})^{p/2}$$

$$\le \ (\text{by Lemma 2.9}) \ K_p \mathsf{E}|X_1|^p(1+\mathsf{E}|X_1|^p)\mathsf{tr}\,(CC^*)^{p/2} \le K_p \mathsf{E}|X_1|^{2p}\mathsf{tr}\,(CC^*)^{p/2}.$$

Therefore, $\mathsf{E}\left|\sum_{i=2}^{n}\overline{X}_i\sum_{j<i}X_j C_{ij}\right|^p$ is bounded by the right side of inequality in Lemma 2.7.

Similarly, $\mathsf{E}\left|\sum_{j=2}^{n}X_j\sum_{i<j}\overline{X}_i C_{ij}\right|^p$ is also bounded by the right side of the inequality, and the proof of the lemma is complete.

REFERENCES

Bai, Z.D., Silverstein, J.W., and Yin, Y.Q. (1988). A note on the largest eigenvalue of a large dimensional sample covariance matrix. *J. Multivariate Anal.* **26** 166-168.

Bai, Z.D. and Yin, Y.Q. (1993) Limit of the smallest eigenvalue of a large dimensional sample covariance matrix *Ann. Probab.* **21** 1275-1294.

Billingsley, P. (1968) *Convergence of Probability Measures.* Wiley, New York.

Burkholder, D.L. (1973). Distribution function inequalities for martingales. *Ann. Probab.* **1** 19-42.

Dilworth, S.J. (1993). Some probabilistic inequalities with applications to functional analysis. *Banach Spaces*, Contemporary Mathematics 144, AMS, Providence 53-67.

Horn, R.A. and Johnson, C.R. (1985). *Matrix Analysis.* Cambridge Univ. Press.

Horn, R.A. and Johnson, C.R. (1991). *Topics in Matrix Analysis.* Cambridge Univ. Press.

Rosenthal, H.P. (1970). On the subspaces of L^p $(p>2)$ spanned by sequences of independent random variables. *Israel J. Math.* 273-303.

Silverstein, J.W. (1995). Strong convergence of the empirical distribution of eigenvalues of large dimensional random matrices *J. Multivariate Anal.* **55** 331-339.

Silverstein, J.W. and Bai, Z.D. (1995). On the empirical distribution of eigenvalues of a class of large dimensional random matrices. *J. Multivariate Anal.* **54** 175-192.

Silverstein, J. W. and Choi S.I. (1995). *Analysis of the limiting spectral distribution of large dimensional random matrices*, J. Multivariate Anal. **54** 295-309.

Silverstein, J.W. and Combettes, P.L. (1992). Signal detection via spectral theory of large dimensional random matrices. *IEEE Trans. Signal Processing* **40** 2100-2105.

Yin, Y.Q., Bai, Z.D., and Krishnaiah, P.R. (1988). On limit of the largest eigenvalue of the large dimensional sample covariance matrix *Probab. Theory Related Fields* **78** 509-521.

Department of Mathematics
National University of Singapore
10 Kent Ridge Crescent
Singapore 119260
E-mail: matbaizd@leonis.nus.sg

Department of Mathematics
Box 8205
North Carolina State University
Raleigh, North Carolina 27695-8205
E-mail: jack@jack.math.ncsu.edu