Analysis of the Limiting Spectral Distribution of Large Dimensional Random Matrices

by

Jack W. Silverstein*

and

Sang-Il Choi
Department of Mathematics, Box 8205
North Carolina State University
Raleigh, North Carolina 27695-8205

Summary

Results on the analytic behavior of the limiting spectral distribution of matrices of sample covariance type, studied in Marčenko and Pastur [2] and Yin [8], are derived. Through an equation defining its Stieltjes transform, it is shown that the limiting distribution has a continuous derivative away from zero, the derivative being analytic wherever it is positive, and resembles $\sqrt{|x-x_0|}$ for most cases of x_0 in the boundary of its support. A complete analysis of a way to determine its support, originally outlined in Marčenko and Pastur [2], is also presented.

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1. Introduction. For $N = 1, 2, \dots$ let $M_N = \frac{1}{N} X_N T_n X_N^*$, where $X_N = \left(X_{ij}^N\right)$ is $N \times n$, $X_{ij}^N \in \mathbb{C}$, identically distributed for all N, i, j, independent across i, j for each N, $\mathsf{E}|X_{1\,1}^1 - \mathsf{E}X_{1\,1}^1|^2 = 1$, $n = n(N)$ with $\frac{n}{N} \to c > 0$ as $N \to \infty$, and T_n is $n \times n$ Hermitian, independent of X_N. For any square matrix A with real eigenvalues, let F^A denote the spectral distribution of A, that is, the empirical distribution function of the eigenvalues of A. Assume $F^{T_n} \xrightarrow{D} H$, a.s., where H is a non-random probability distribution function (p.d.f.). Then it is known that, with probability one, $F^{M_N} \xrightarrow{D} F$, a non-random p.d.f. depending on H and c, if either: 1) T_n is diagonal ([2],[4]), or 2) $T_n \geq 0$ (T_n is non-negative definite), and H has moments of all order satisfying Carleman's sufficiency condition (ensuring only one p.d.f. having these moments) ([8]).

This result has direct bearing on multivariate statistical applications when the vector dimension and sample size are both large but have the same order of magnitude (see [6] for an application to array signal processing). Indeed, when $T_n \geq 0$ and $\mathsf{E}X_{1\,1}^1 = 0$, the matrix $\frac{1}{N} T_n^{1/2} X_N^* X_N T_n^{1/2}$ ($T_n^{1/2}$ being any Hermitian square root of T_n) can be viewed as a sample covariance matrix formed from N samples of the random vector $T_n^{1/2}(X_N^*)_{\cdot\,1}$. The spectrum of this matrix agrees with that of $\frac{1}{N} T_n X_N^* X_N$, and for any Hermitian T_n the spectra of this latter matrix and M_N differ by $|n - N|$ zero eigenvalues. From this it is a simple matter to verify $F^{M_N} = (1 - \frac{n}{N}) 1_{[0,\infty)} + \frac{n}{N} F^{\frac{1}{N} T_n X_N^* X_N}$ (1_B denoting the indicator function on the set B). Thus, almost surely,

$$F^{\frac{1}{N} T_n X_N^* X_N} \xrightarrow{D} F_0 \equiv (1 - \frac{1}{c}) 1_{[0,\infty)} + \frac{1}{c} F. \tag{1.1}$$

Important to applications is the behavior of F and its dependence on H and c. The purpose of this paper is to derive certain fundamental properties, the most important being the analyticity of F.

Under condition 2) it is shown in [8] that F_0 has moments of all order satisfying Carleman's sufficiency condition, and are explicitly expressed. From the moments, F_0 has been derived in two cases: when $T_n = I_n$ (the $n \times n$ identity matrix), that is, when $H = 1_{[1,\infty)}$ ([1]), and when $T_n = (\frac{1}{m} Y_n Y_n^T)^{-1}$, where Y_n is $n \times m$ with $n < m$, $n/m \to y \in (0, 1)$, and contains i.i.d. $N(0, 1)$ entries ([3]). In both cases F_0 has a continuous density on $(0, \infty)$. The moments of F_0 can also yield some qualitative behavior ([5]), namely: *i)* c and F_0 uniquely determine H, and *ii)* $F_0 \xrightarrow{D} H$ as $c \to 0$ (which should be expected, since by the law of large numbers $\frac{1}{N} T_n X_N^* X_N \xrightarrow{i.p.} T_n$ for n fixed and $N \to \infty$).

Further analysis relying on the moment expressions appear extremely difficult. However, the approach taken in [2],[4] under condition 1) (where H is arbitrary) leads to a characterization of F most suitable to analysis. It uses the Stieltjes transforms of measures, that is, for $z \in D \equiv \{z \in \mathbb{C} : Im\, z > 0\}$ and p.d.f. G on $\mathbb{R}$, the Stieltjes transform m_G of G is the analytic function mapping D into itself defined by $m_G(z) = \int \frac{dG(\lambda)}{\lambda - z}$ for $z \in D$. Because of the well-known inversion formula

$$G\{[a,b]\}=\lim_{\eta\to 0^+}\frac{1}{\pi}\int_a^b Im\, m_G(\xi+i\eta)d\xi \tag{1.2}$$

(a,b continuity points of G), p.d.f.'s are uniquely determined by their Stieltjes transform.

It is shown in [2],[4] that:

For each $z\in D$, $m=m_F(z)$ is the unique solution for $m\in D$ to the equation

$$m=-\left(z-c\int\frac{\lambda dH(\lambda)}{1+\lambda m}\right)^{-1}. \tag{1.3}$$

It follows that:

On D, $m_F(z)$ has a unique inverse, given by

$$z_F(m)=-\frac{1}{m}+c\int\frac{\lambda dH(\lambda)}{1+\lambda m}\qquad m\in m_F(D). \tag{1.4}$$

Although it appears unlikely a general form for F exists, quite a bit can be inferred from this representation. We show how the above qualitative properties can be derived. From (1.4) we find for all $m\in m_F(D)$ $z_F(m)=-\frac{1}{m}+\frac{c}{m}-\frac{c}{m^2}m_H(-\frac{1}{m})$. Thus, m_H is determined by F or F_0 (via z_F) and c on the open set $\{m:-\frac{1}{m}\in m_F(D)\}$. Therefore $i)$ follows. For the second property, we fix $z\in D$. From (1.1) we have $m_F(z)=-\frac{(1-c)}{z}+cm_{F_0}(z)$. Using the fact that $|m_G(z)|\le\frac{1}{Im\,z}$ for any p.d.f. G, we find $m_F(z)\to-\frac{1}{z}$ as $c\to 0$. From (1.4) it follows that $zm_{F_0}(z)=-1+\int\frac{\lambda m_F(z)}{1+\lambda m_F(z)}dH(\lambda)$. Since $\left|\frac{\lambda m_F(z)}{1+\lambda m_F(z)}\right|\le\frac{|m_F(z)|}{Im\,m_F(z)}$, which is bounded due to the convergence of $m_F(z)$, from the dominated convergence theorem (d.c.t.), we conclude as $c\to 0$ $zm_{F_0}(z)\longrightarrow -1-\int\frac{\lambda/z}{1-\lambda/z}dH(z)=zm_H(z)$. Thus we get $ii)$.

We can also use (1.4) to show $F\{0\}$, the mass F places at 0, is $\max(0,1-c(1-H\{0\}))$. Consider a sequence $\{T_n\}$ from 1) satisfying $F^{T_n}\xrightarrow{D}H$, and $F^{T_n}\{0\}\to H\{0\}$. Then it is a simple matter to verify $F^{M_N}\{0\}\ge\max(0,\frac{N-(n-nF^{T_n}\{0\})}{N})$, which implies $F\{0\}\ge\max(0,1-c(1-H\{0\}))$. For any p.d.f. G it is straightforward to show $\lim_{y\downarrow 0}iym_G(iy)=-G\{0\}$. Then, from (1.4) we find $F\{0\}=1-c+c\lim_{y\downarrow 0}\int\frac{dH(\lambda)}{1+\lambda m_F(iy)}$. If $F\{0\}>0$, then, as $y\downarrow 0$, $|m_F(iy)|\to\infty$, and, since $yRe\,m_F(iy)\to 0$ and $yIm\,m_F(iy)\to F\{0\}$, we have $\frac{Re\,m_F(iy)}{Im\,m_F(iy)}\to 0$ as $y\downarrow 0$. Using $\frac{1}{|1+\lambda m_F(iy)|^2}\le 1+\left(\frac{Re\,m_F(iy)}{Im\,m_F(iy)}\right)^2$ and the d.c.t., we conclude $F\{0\}=1-c+c\int 1_{\{0\}}(\lambda)dH(\lambda)=1-c+cH\{0\}$. Thus, $F\{0\}=\max(0,1-c(1-H\{0\}))$.

Other properties previously derived from (1.3) include the continuity of F away from 0 ([7],[5]), a method for determining S_F, the support of F, and the behavior of F near certain points in ∂S_F, the boundary of S_F ([2]). The latter two will be given full treatment in this paper. The main goal is to establish the following result.

Theorem 1.1. For all $x\in\mathbb{R}$, $x\ne 0$,

$$\lim_{z\in D\to x}m_F(z)\equiv m_0(x)\quad\text{ exists.} \tag{1.5}$$

The function m_0 is continuous on $\mathbb{R}-\{0\}$. Consequently (see Theorem 2.1 below), F has a continuous derivative f on $\mathbb{R}-\{0\}$ given by $f(x)=\frac{1}{\pi}Im\,m_0(x)$. The density f is analytic (possesses a power series expansion) for every $x \neq 0$ for which $f(x) > 0$. Moreover, for these x, $\pi f(x)$ is the imaginary part of the unique $m \in D$ satisfying

$$x = -\frac{1}{m} + c\int \frac{\lambda dH(\lambda)}{1+\lambda m}. \tag{1.6}$$

Obviously the theorem reveals much of the analytic behavior of F in general, and its dependence on H. For example, when H is discrete with finite support, $m_0(x)$ is the root of a polynomial with coefficients depending linearly on x, making f algebraic in nature. The theorem also shows how to determine F. For some H (1.6) can be solved explicitly, for example, in the above 2 cases, or when H has support on at most 3 distinct points in $\mathbb{R}$ (the degree of the resulting polynomial being at most 4). If no way of solving (1.6) is apparent, then a simple numerical scheme can be applied.

It is remarked here that, even if $F\{0\} = 0$, it is still possible for f not to exist at 0. For example, for the case $H = 1_{[1,\infty)}$ and $c = 1$, $f(x) = 1_{(0,4)}\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$ ([1]).

The proof of Theorem 1.1 relies on a result concerning the existence of a derivative of a p.d.f. whenever the imaginary part of its Stieltjes transform converges. It will be stated and proven in the next section, along with a result needed to establish the continuity of m_0. The third section completes the proof of Theorem 1.1. The fourth section gives a detailed analysis on how the support of F can be determined from the graph of (1.6) on $\{m \in \mathbb{R} : m \neq 0, -\frac{1}{m} \in S_H^c\}$. Section 5 shows, for most cases of $x_0 \in \partial S_F$, that $f(x)$, for $x \in S_F$, resembles $\sqrt{|x - x_0|}$ near x_0. Section 6 contains some concluding remarks.

2. Preliminary Results. The following theorems are stated under conditions sufficient for this paper. Weaker conditions clearly exist for both.

Theorem 2.1 Let G be a p.d.f. and $x_0 \in \mathbb{R}$. Suppose $Im\, m_G(x_0) \equiv \lim_{z\in D\to x_0} Im\, m_G(z)$ exists. Then G is differentiable at x_0, and its derivative is $\frac{1}{\pi} Im\, m_G(x_0)$.

Proof. Fix $\epsilon > 0$. Let $\delta > 0$ be s.t. $\frac{1}{\pi}|Im\, m_G(x+iy) - Im\, m_G(x_0)| < \frac{\epsilon}{2}$ whenever $|x - x_0| < \delta$, $y \in (0, \delta)$. Let $x_1 < x_2$ be continuity points of G s.t. $x_1 < x_2$ and $|x_i - x_0| < \delta$, $i = 1, 2$. From (1.2), we can choose $y \in (0, \delta)$ s.t. $\left|G(x_2) - G(x_1) - \frac{1}{\pi}\int_{x_1}^{x_2} Im\, m_G(x+iy)dx\right| < \frac{\epsilon}{2}(x_2 - x_1)$. For any $x \in [x_1, x_2]$, we have $|x - x_0| < \delta$. Thus

$$\left|\frac{G(x_2) - G(x_1)}{x_2 - x_1} - \frac{1}{\pi} Im\, m_G(x_0)\right| \leq \frac{1}{x_2 - x_1}\left|G(x_2) - G(x_1) - \frac{1}{\pi}\int_{x_1}^{x_2} Im\, m_G(x+iy)dx\right|$$
$$+ \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}\left|\frac{1}{\pi}\left(Im\, m_G(x+iy) - Im\, m_G(x_0)\right)\right| dx < \epsilon.$$

It follows that G is continuous at x_0, and for any sequence $\{x_n\}$ of continuity points of G converging to x_0

$$\lim_{n\to\infty} \frac{G(x_n) - G(x_0)}{x_n - x_0} = \frac{1}{\pi} Im\, m_G(x_0). \tag{2.1}$$

Let $\{x_n\}$ be any real sequence satisfying $x_n \downarrow x_0$. For each n choose continuity points $x_{cp}^{(n)-}, x_{cp}^{(n)+}$ s.t. $x_0 < x_{cp}^{(n)-} \leq x_n \leq x_{cp}^{(n)+}$, $(1 - \frac{1}{n})(x_n - x_0) \leq x_{cp}^{(n)-} - x_0$, and

$x_{cp}^{(n)+} - x_0 \le (1 + \frac{1}{n})(x_n - x_0)$. Then

$$\left(1 - \frac{1}{n}\right) \frac{G(x_{cp}^{(n)-}) - G(x_0)}{x_{cp}^{(n)-} - x_0} \le \frac{G(x_n) - G(x_0)}{x_n - x_0} \le \left(1 + \frac{1}{n}\right) \frac{G(x_{cp}^{(n)+}) - G(x_0)}{x_{cp}^{(n)+} - x_0},$$

and we have (2.1) holding for this sequence. A similar argument can be made for $\{x_n\}$ with $x_n \uparrow x_0$. This complete the proof. $\square$

Theorem 2.2. Let X be an open and bounded subset of $\mathbb{R}^n$, let Y be an open and bounded subset of $\mathbb{R}^m$, and let $f : \overline{X} \to Y$ be a function, continuous on X. If, for all $x_0 \in \partial X$, $\lim_{x \in X \to x_0} f(x) = f(x_0)$, then f is continuous on all of $\overline{X}$.

Proof. Let $x_0 \in \partial X$. Given $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in X$, $\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \frac{\epsilon}{2}$ ($\| \cdot \|$ denoting Euclidean norm). Let $\hat{x} \in \partial X$ satisfy $\|\hat{x} - x_0\| < \delta$. Then, since there exists $\bar{x} \in X$ such that $\|\bar{x} - x_0\| < \delta$ and $\|f(\hat{x}) - f(\bar{x})\| < \frac{\epsilon}{2}$, we must have $\|f(\hat{x}) - f(x_0)\| < \epsilon$. Thus, f is continuous for all $x \in \overline{X}$. $\square$

Once (1.5) is verified, existence of the continuous density f on $\mathbb{R} - \{0\}$ will follow from the above theorems. Clearly, showing $\lim_{z \in D \to x} Im\, m_F(z) = Im_0(x)$ $(x \ne 0)$ is sufficient, since the mapping $z \to Im\, z$ is continuous. As will be seen (in the next section), the latter is verified mid-way through the proof. The importance in establishing (1.5) lies mainly in analyzing the behavior of f at boundary points of its support, to be discussed in Section 5.

3. Proof of Theorem 1.1. For $z \in D$ we write $z = x + iy$, and $m_F(z) = m_1 + im_2$. The open ball in $\mathbb{C}$ with radius r centered at z will be denoted by $B(z, r)$. From (1.4) we find the following relationship between (x, y) and (m_1, m_2)

$$\begin{aligned} x &= -\frac{m_1}{m_1^2 + m_2^2} + c \int \frac{\lambda(1 + \lambda m_1) dH(\lambda)}{(1 + \lambda m_1)^2 + \lambda^2 m_2^2} \\ y &= m_2 \left(\frac{1}{m_1^2 + m_2^2} - c \int \frac{\lambda^2 dH(\lambda)}{(1 + \lambda m_1)^2 + \lambda^2 m_2^2} \right). \end{aligned} \tag{3.1}$$

Lemma 3.1. $m_2(x + iy)$ is bounded for $x + iy$ lying in any bounded region of D away from the imaginary axis.

Proof. Suppose not. Then there exists a sequence $\{(x_n, y_n)\}$ such that $x_n \to \bar{x} \ne 0$, and $m_2^n \equiv m_2(x_n + iy_n) \uparrow +\infty$ as $n \to \infty$. Let $m_1^n = m_1(x_n + iy_n)$. We have

$$x_n = -\frac{m_1^n}{(m_1^n)^2 + (m_2^n)^2} + c \int \frac{\lambda(1 + \lambda m_1^n) dH(\lambda)}{(1 + \lambda m_1^n)^2 + \lambda^2 (m_2^n)^2} \to \bar{x} \text{ as } n \to \infty.$$

However, using $|ab| \le \frac{1}{2}(a^2 + b^2)$, we find

$$\max \left(\left| \frac{m_1^n}{(m_1^n)^2 + (m_2^n)^2} \right|, \left| \frac{\lambda(1 + \lambda m_1^n)}{(1 + \lambda m_1^n)^2 + \lambda^2 (m_2^n)^2} \right| \right) \le \frac{1}{2 m_2^n} \to 0 \text{ as } n \to \infty.$$

Therefore, by the d.c.t., we have $\bar{x}_n \to 0$, a contradiction. $\square$

Lemma 3.2. Under the same conditions as in Lemma 3.1, $m_1(x+iy)$ is bounded.

Proof. We prove again by contradiction, using the same notation as above, where now $|m_1(x_n+i\bar{y}_n)| \uparrow \infty$ as $n\to\infty$. We have $\frac{|m_1^n|}{(m_1^n)^2+(m_2^n)^2} \le \frac{1}{|m_1^n|} \to 0$ as $n\to\infty$. Since $y_n>0$ and $m_2^n>0$ for all n, we have, from (3.1), $\frac{1}{(m_1^n)^2+(m_2^n)^2} > c\int \frac{\lambda^2 dH(\lambda)}{(1+\lambda m_1^n)^2+\lambda^2(m_2^n)^2}$. Therefore, $\forall\, \epsilon\in[0,1)$, we have

$$c\int \frac{\lambda^2|m_1^n|^{1+\epsilon}dH(\lambda)}{(1+\lambda m_1^n)^2+\lambda^2(m_2^n)^2} \le \frac{|m_1^n|^{1+\epsilon}}{(m_1^n)^2+(m_2^n)^2} \le \frac{1}{|m_1^n|^{1-\epsilon}} \to 0,$$

which implies $\lim_{n\to\infty}\int \frac{\lambda^2 m_1^n dH(\lambda)}{(1+\lambda m_1^n)^2+\lambda^2(m_2^n)^2} = 0$. Moreover, for $\epsilon\in(0,1)$, we have as $n\to\infty$

$$\int_{\mathbb{R}-\left[-\frac{1}{|m_1^n|^{1+\epsilon}},\frac{1}{|m_1^n|^{1+\epsilon}}\right]} \frac{|\lambda|dH(\lambda)}{(1+\lambda m_1^n)^2+\lambda^2(m_2^n)^2} \le \int \frac{\lambda^2|m_1^n|^{1+\epsilon}dH(\lambda)}{(1+\lambda m_1^n)^2+\lambda^2(m_2^n)^2} \to 0.$$

It is easy to verify that $\frac{\lambda}{(1+\lambda m_1)^2+\lambda^2 m_2^2}$ is increasing on $[-(m_1^2+m_2^2)^{-\frac12},(m_1^2+m_2^2)^{-\frac12}]$, and, since m_2^n is bounded, $\frac{1}{|m_1^n|^{1+\epsilon}} < ((m_1^n)^2+(m_2^n)^2)^{-\frac12}$, for n sufficiently large. Therefore, for n large,

$$\int_{\left[-\frac{1}{|m_1^n|^{1+\epsilon}},\frac{1}{|m_1^n|^{1+\epsilon}}\right]} \frac{|\lambda|dH(\lambda)}{(1+\lambda m_1^n)^2+\lambda^2(m_2^n)^2} \le \frac{\frac{1}{|m_1^n|^{1+\epsilon}}}{\left(1-\frac{|m_1^n|}{|m_1^n|^{1+\epsilon}}\right)^2} \to 0 \text{ as } n\to\infty.$$

Therefore $\lim_{n\to\infty}\int\frac{\lambda dH(\lambda)}{(1+\lambda m_1^n)^2+\lambda^2(m_2^n)^2}=0$, and we conclude as in the previous lemma that $x_n\to 0$, a contradiction. □

Fix an $x_0\in\mathbb{R}-\{0\}$. Because of the two lemmas (1.5) follows if convergent subsequences of $\{m_F(z_n)\}$ for any $\{z_n\}\subset D$, $z_n\to x_0$, are shown to be unique. Since m_F is analytic on $\mathbb{C}-S_F$, (1.5) holds for all $x\in S_F^c$. Thus we may assume $x_0\in S_F$.

Lemma 3.3. If there exists a sequence $\{\bar{z}_n\}\subset D$ for which $\lim_{\bar{z}_n\to x_0} m_F(\bar{z}_n)=\bar{m}\in D$ then $\lim_{z\in D\to x_0} m_F(z)$ exists.

Proof. Let $\bar{m}_n = m_F(\bar{z}_n)$. From (1.4), we have $\bar{z}_n = z_F(\bar{m}_n)$, and since $\int\frac{\lambda dH(\lambda)}{1+\lambda m}$ is analytic for $m\in D$, we have $z_F(\bar{m})=x_0$ (z_F considered as an analytic function on $\mathbb{C}-\{m : m=0 \text{ or } -m^{-1}\in S_H\}$).

Let $B=B(\bar{m},\epsilon)$ where $0<\epsilon<Im\,\bar{m}$. Then $z_F(m)$ is analytic and nonconstant on B. Therefore, by the open mapping theorem (o.m.t.), $z_F(B)$ is open and contains x_0. Thus, for any $\{z_n\}\subset D$ with $z_n\to x_0$, $z_n\in z_F(B)$ for large n. For these z_n, there exists $m_n\in B$ such that $z_F(m_n)=z_n$. But $B\subset D$, so that by the uniqueness property in (1.3), we must have $m_F(z_n)=m_n$ which lies in B. Therefore, since ϵ was arbitrary, $\lim_{z\in D\to x_0} m(z)$ exists and the limit must be $\bar{m}$. □

At this stage we know $\lim_{z\in D\to x_0} Im\, m_F(z)$ converges. To complete the proof of (1.5), we need to show convergence when

$$\lim_{z\in D\to x_0} Im\, m_F(z) = 0. \tag{3.2}$$

Lemma 3.4. For fixed $a \in \mathbb{R}$ satisfying $\frac{1}{a^2+t} - c\int \frac{\lambda^2 dH(\lambda)}{(1+\lambda a)^2+\lambda^2 t} \neq 0$ for some $t > 0$, there are at most two positive t's satisfying $\frac{1}{a^2+t} - c\int \frac{\lambda^2 dH(\lambda)}{(1+\lambda a)^2+\lambda^2 t} = 0$.

Proof. We can assume $H \neq 1_{[0,\infty)}$. Consider $g(t) = (a^2+t)\int \frac{\lambda^2 dH(\lambda)}{(1+\lambda a)^2+\lambda^2 t}$ for $t > 0$. It can easily be verified that g is analytic on $(0,\infty)$, and we can assume it is non-constant. For positive integer n, let $d_n(t) = \int \frac{\lambda^{2n} dH(\lambda)}{((1+\lambda a)^2+\lambda^2 t)^n}$. The n^{th} order derivative can be expressed as $(-1)^{n+1} n!(d_n(t) - (a^2+t)d_{n+1}(t))$. It follows that for any local extreme point t the first non-zero n^{th} order derivative, where necessarily, n must be even, can be written as $(-1)^{n+1} n! d_n^{-1}(t)(d_n^2(t) - d_{n-1}(t)d_{n+1}(t))$ which, by a simple application of the Schwarz inequality, must be positive. Thus g can have at most one local extreme, and consequently, there are at most two solutions to $g(t) = c^{-1}$. This completes the proof. □

Lemma 3.5. Assume $0 \notin (m_a, m_b) \subset \mathbb{R}$. If $\int \frac{h\lambda^2 dH(\lambda)}{(1+\lambda m_0)^2+\lambda^2 h^2} \to 0$ as $h \downarrow 0$, $\forall\, m_0 \in (m_a, m_b)$, then $(-m_a^{-1}, -m_b^{-1}) \subset S_H^c$.

Proof. Fix $m_0 \in (m_a, m_b)$. Let $g(\lambda) = \frac{h\lambda^2}{(1+\lambda m_0)^2+\lambda^2 h^2}$. It is a simple matter to show for $0 < h < |m_0^{-1}|$, g is increasing on $A \equiv (-m_0^{-1} - h, -m_0^{-1}]$ and is decreasing on $(-m_0^{-1}, -m_0^{-1}+h]$. We have $\int g(\lambda)dH(\lambda) \geq$

$$\int_A \frac{h(m_0^{-1}+h)^2 dH(\lambda)}{(hm_0)^2 + h^2(m_0^{-1}+h)^2} = \frac{(m_0^{-1}+h)^2}{m_0^2 + (m_0^{-1}+h)^2} \frac{H(-m_0^{-1}) - H(-m_0^{-1}-h)}{h}.$$

Thus, $\lim_{h\downarrow 0} \frac{H(-m_0^{-1})-H(-m_0^{-1}-h)}{h} = 0$. Similarly, we find $\lim_{h\downarrow 0} \frac{H(-m_0^{-1}+h)-H(-m_0^{-1})}{h} = 0$. Therefore $H'(-m_0^{-1})$ exists and is 0. Since m_0 was arbitrary on (m_a, m_b), H is constant on $(-m_a^{-1}, -m_b^{-1})$. Therefore $(-m_a^{-1}, -m_b^{-1}) \subset S_H^c$. □

Lemma 3.6. Assume (3.2). As $n \to \infty$, if $z_n \in D \to x_0$, $\hat{z}_n \in D \to x_0$, $m_F(z_n) \to m_0 \in \mathbb{R}$, $m_F(\hat{z}_n) \to \hat{m}_0 \in \mathbb{R}$ and $m_0 < \hat{m}_0$, then, $\forall\, \bar{m} \in (m_0, \hat{m}_0)$, there exists $\{\bar{z}_n\} \subset D$ such that $\bar{z}_n \to x_0$, and $m_F(\bar{z}_n) \to \bar{m}$. The sequence $\{\bar{z}_n\}$ can be chosen so that $Re\, m_F(\bar{z}_n) = \bar{m}$.

Proof. Fix $\bar{m} \in (m_0, \hat{m}_0)$. For all n sufficiently large, there exists $\bar{z}_n$ on the line segment joining z_n and $\hat{z}_n$ s.t. $Re\, m_F(\bar{z}_n) = \bar{m}$. Necessarily, $\bar{z}_n \to x_0$, so that $Im\, m_F(\bar{z}_n) \to 0$. □

Lemma 3.7. Under the same assumptions as in Lemma 3.6, and, additionally, $0 \notin (m_0, \hat{m}_0)$, we have $(-m_0^{-1}, -\hat{m}_0^{-1}) \subset S_H^c$.

Proof. Fix $m_1 \in (m_0, \hat{m}_0)$. From Lemma 3.6 we know that for any $\delta > 0$, $m_F(D)$ intersects the line $\{m_1 + im_2 : \delta \geq m_2 > 0\}$ at infinitely many points. Therefore, for these points, from (3.1), we have

$$\frac{1}{m_1^2+m_2^2} - c\int \frac{\lambda^2 dH(\lambda)}{(1+\lambda m_1)^2+\lambda^2 m_2^2} > 0. \tag{3.4}$$

From Lemma 3.4, $\{m_2 > 0 : \frac{1}{m_1^2+m_2^2} - c\int \frac{\lambda^2 dH(\lambda)}{(1+\lambda m_1)^2+\lambda^2 m_2^2} = 0\}$ has at most two points. Therefore, $\exists\, \delta_0 > 0$, such that $\forall\, m_2 \in (0, \delta_0]$, (3.4) holds, which implies $\int \frac{m_2\lambda^2 dH(\lambda)}{(1+\lambda m_1)^2+\lambda^2 m_2^2} \to$

0 as $m_2 \downarrow 0$. Thus, from Lemma 3.5, $(-m_0^{-1}, -\hat{m}_0^{-1}) \subset {S_H}^c$. □

We can now complete the proof of (1.5). Suppose (3.2) holds but $m(z)$ does not converge as $z \to x_0$. Because of Lemmas 3.1,3.2, the conditions of Lemma 3.6 are satisfied and we can find $(m_0, \hat{m}_0)$ such that $0 \notin (m_0, \hat{m}_0)$. Therefore, from Lemma 3.7, $(-m_0^{-1}, -\hat{m}_0^{-1}) \subset {S_H}^c$. Moreover, from Lemma 3.6, for any $\bar{m} \in (m_0, \hat{m}_0)$, there exists $\{\bar{z}_n\} \subset D$ such that $\bar{z}_n \to x_0$ and $m_F(\bar{z}_n) \to \bar{m}$ as $n \to \infty$. Since z_F is the inverse of m_F on $m_F(D)$ and analytic on $\mathbb{C} - \{m : m = 0 \text{ or } -m^{-1} \in S_H\})$, we have $x_0 = z_F(\bar{m})$, $\forall\, \bar{m} \in (m_0, \hat{m}_0)$.

Therefore, z_F is a constant function on $(m_0, \hat{m}_0)$, and, consequently, on $m_F(D)$, a contradiction. Therefore, (1.5) holds for all $x \in \mathbb{R} - \{0\}$.

The existence of the density f, defined in terms of m_0, and continuous on $x \in \mathbb{R} - \{0\}$ follow from the theorems in Section 2. Continuity gives us $\pi f(x)$ being the imaginary part of a solution to (1.6) when $f(x) > 0$. The following argument shows uniqueness. Suppose distinct $m, m' \in D$ satisfy (1.6). There exist disjoint open balls $B(m, \epsilon), B(m', \epsilon')$ each lying entirely in D. As a result of the o.m.t., $z_F(B(m, \epsilon)) \cap z_F(B(m', \epsilon'))$ is open and contains x. Therefore, there exists a $z \in D$ and distinct $\underline{m}, \underline{m}' \in D$ such that both $(z, \underline{m})$ and $(z, \underline{m}')$ satisfy the equation in (1.3), violating the uniqueness of the solution.

It remains to show the analyticity of f. Consider an $x_0 \in \mathbb{R} - \{0\}$ for which $f(x_0) > 0$. We can eliminate the case $H = 1_{[0,\infty)}$ since all mass at zero will yield $F = 1_{[0,\infty)}$ as well, resulting in $f \equiv 0$ on $\mathbb{R} - \{0\}$. Let $m = m_0(x_0) = m_1 + im_2$. We have $m_2 > 0$, and therefore, z_F is analytic at m. Moreover, from continuity, $z_F(m) = x_0$.

Suppose $z_F'(m) = 0$. By splitting up the real and imaginary parts, this yields two equations in m_1 and m_2. Another equation arises from the imaginary part of $z_F(m)$ being zero. We arrive at a 3×3 linear homogeneous system with unknowns $\frac{1}{|m|^4} - c \int \frac{\lambda^4 dH(\lambda)}{|1+\lambda m|^4}$, $c \int \frac{\lambda^3 dH(\lambda)}{|1+\lambda m|^4}$, $c \int \frac{\lambda^2 dH(\lambda)}{|1+\lambda m|^4}$, and non-singular coefficient matrix. This implies $\int \frac{\lambda^2 dH(\lambda)}{|1+\lambda m|^4} = 0$, which can only hold if $H = 1_{[0,\infty)}$. Therefore, $z_F'(m) \neq 0$, so that by the inverse function theorem (i.f.t.), in a neighborhood $U \subset D$ of m, z_F has an inverse on $z_F(U)$, which contains x_0. This inverse must agree with m_F on $z_F(U) \cap D$. Therefore, m_F has an analytic extension onto $z_F(U)$. Thus, near x_0 $m_F(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n$ and therefore, $f(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} Im\, a_n\, (x - x_0)^n$ for any real x near x_0.

This completes the proof of Theorem 1.1.

4. The Support of F. It is mentioned in [2] that, on open intervals in S_F^c, the function $m_F(x)$ exists, and is continuous, real and increasing. Therefore, the inverse function exists on these intervals, also being continuous, real and increasing. These statements hold true for any Stieltjes transform. By finding the intervals on which the inverse is increasing and computing their values on these intervals, S_F can be determined. Intuitively, (1.6) must be the inverse, and [2] make that claim, however, without proof. Several things need to shown. For example, even though the domain of (1.6) is clearly

$B \equiv \{m \in \mathbb{R} : m \neq 0, -m^{-1} \in S_H^c\}$, might there exist an extension of this function into B^c (in $\mathbb{R}$) which constitutes part of the inverse of m_F? That is, in order to determine S_F, do we need to look further than simply plotting (1.6) on B? As will be shown the answer is no. Plot (1.6) on B. Find the extreme values on each increasing interval. Delete these points and everything in between on the real line. Do this for all increasing intervals. What is left will be S_F.

The above fact will follow from the first two theorems. This section also contains general qualitative information on the graph of (1.6) useful for applications (see [6]). For example, it will be shown that each interval in B corresponds to at most one interval in S_F^c

In the following, x_F will denote (1.6), that is, the restriction of z_F on B.

Theorem 4.1 For any $x_0 \in I \subset S_F^c$, where I is an open interval, let $m_0 = m_F(x_0)$. Then $m_0 \in B$, $x_0 = x_F(m_0)$, and $x_F'(m_0) = \frac{1}{m_0^2} - c\int \frac{\lambda^2 dH(\lambda)}{(1+\lambda m_0)^2} > 0$.

Proof. We have $m_F'(x_0) = \int \frac{dF(\lambda)}{(\lambda - x_0)^2} > 0$. Thus, by the i.f.t., m_F has an inverse $\hat{z}(m)$ in a neighborhood N of x_0. By the o.m.t., $m_F(N)$ is open and contains m_0. Therefore, for $m \in m_F(N) \to m_0$, we have $\hat{z}(m) \to x_0$. But, on $m_F(N \cap D) = m_F(N) \cap D$, from (1.3), we must have $\hat{z} = z_F$, that is, $\hat{z}$ extends z_F analytically onto $m_F(N)$. Therefore, for $m \in D \to m_0$, we have $z_F(m) \to x_0$.

Suppose that $m_0 = 0$. For $m = i\bar{m}$, $\bar{m} > 0$, we have $Im\, z_F(i\bar{m}) \to 0$ as $\bar{m} \downarrow 0$, and, therefore, from (3.1) $\frac{1}{\bar{m}}(1 - c\int \frac{\bar{m}^2\lambda^2 dH(\lambda)}{1+\bar{m}^2\lambda^2}) \to 0$. By the d.c.t. we have $\int \frac{\bar{m}^2\lambda^2 dH(\lambda)}{1+\bar{m}^2\lambda^2} \to 0$ as $\bar{m} \downarrow 0$. Thus $\bar{m}^{-1}$ must converge to 0, a contradiction. Therefore, $m_0 \neq 0$.

From (1.4) it is straightforward to verify $m_H(m) = \frac{-z_F(-m^{-1}) + m(1-c)}{cm^2}$. Therefore, as $m \in D \to -m_0{}^{-1}$, we have $m_H(m)$ converging to a real number. Thus, by Theorem 2.1, $H'(-m_0{}^{-1})$ exists and is equal to 0. This implies $H' = 0$ on $J \equiv \{-\frac{1}{m_F(x)} : x \in I\}$, which is open due to the monotonicity of m_F on I. Thus H is constant on J, implying $J \subset S_H^c$, so that x_F is well-defined at m_0, and we have $x_0 = \hat{z}(m_0) = x_F(m_0)$. Since $m_F'(x_0) > 0$ we have $x_F'(m_0) > 0$, and the proof is complete. □

Theorem 4.2. Suppose that $m_0 \in B$ and $x_F'(m_0) = \frac{1}{m_0^2} - c\int \frac{\lambda^2 dH(\lambda)}{(1+\lambda m_0)^2} > 0$. Let $x_0 = x_F(m_0)$. Then $x_0 \in S_F^c$ and $m_F(x_0) = m_0$.

Proof. By the i.f.t. z_F has an inverse $\hat{m}(z)$ in a neighborhood $B(m_0, \delta)$ of m_0 for some $\delta > 0$ satisfying: $0 \notin (m_0 - \delta, m_0 + \delta)$; $\forall\, m \in (m_0 - \delta, m_0 + \delta)$, $-m^{-1} \in S_H{}^c$; and $\forall\, m = m_1 + im_2 \in B(m_0, \delta)$ (3.4) holds. By the o.m.t., $z_F(B(m_0, \delta))$ is open and $x_0 \in z_F(B(m_0, \delta))$. Thus, for some $\epsilon > 0$, $(x_0 - \epsilon, x_0 + \epsilon) \subset z_F(B(m_0, \delta))$. Notice, for $m \in B(m_0, \delta)$, from (3.1) and (3.4), $z_F(m) \in \mathbb{R} \iff m \in \mathbb{R}$. Therefore, for $z \in z_F(B(m_0, \delta) \to x \in (x_0 - \epsilon, x_0 + \epsilon)$, we must have $\hat{m}(z)$ converging to a real number.

From (3.1) and (3.4), we have $z_F(B(m_0, \delta) \cap D) \subset D$. From (1.3), we must have $m_F(z_F(B(m_0, \delta) \cap D)) = B(m_0, \delta) \cap D$ and $m_F(z) = \hat{m}(z)$ for any $z \in z_F(B(m_0, \delta) \cap D)$. Therefore, $\hat{m}$ extends m_F analytically onto $z_F(B(m_0, \delta))$, and for any $z \in D \to x \in$

$(x_0-\epsilon, x_0+\epsilon)$, we must have $Im\, m_F(z) \to 0$. Therefore, by Theorem 2.1, $F'(x)$ exists and is 0. Thus F is constant on $(x_0-\epsilon, x_0+\epsilon)$, which implies $x_0 \in S_F^c$. Therefore, m_F is well defined at x_0, and, consequently, $m_F(x_0) = m_0$. □

From these two theorems we can conclude: $x \in S_F^c \Longleftrightarrow m \in B$ and $x_F'(m) > 0$ (in either direction, x and m are real and are related by $m_F(x) = m$ and $x_F(m) = x$).

Each of the remaining theorems in this section sheds some light on the qualitative behavior of the graph of x_F, useful in determining S_F. For ease of exposition, we assume $H \neq 1_{[0,\infty)}$.

Theorem 4.3. Suppose $m_1 < m_2$, $[m_1, m_2] \subset B$ and $x_F'(m_i) \geq 0$ for $i = 1, 2$. Then $x_F'(m) > 0$ for all $m \in (m_1, m_2)$.

Proof. Write $x_F'(m) = m^{-2}(1 - cg(m))$ where $g(m) = \int \frac{\lambda^2 m^2 dH(\lambda)}{(1+\lambda m)^2}$. Suppose that, for some $\bar{m} \in (m_1, m_2)$, $x_F'(\bar{m}) \leq 0$. Then it follows that $g(m)$ has a local maximum at some $m_0 \in (m_1, m_2)$. However, upon computing the second derivative of g we find that $g''(m_0) = 6 \int \frac{\lambda^2 dH(\lambda)}{(1+\lambda m_0)^4} > 0$, a contradiction. □

Thus, on any interval $I \subset B$, there is at most one interval on which $x_F(m)$ is increasing, in the sense that there cannot be two disjoint intervals in I on which $x_F(m)$ is increasing separated by either an interval on which $x_F(m)$ is nonincreasing, or by a point where $x(m)$ has zero derivative.

Theorem 4.4. Let $[m_1, m_2]$, $[m_3, m_4]$ be two disjoint intervals in B satisfying $\forall\, m \in (m_1, m_2) \cup (m_3, m_4)$, $x_F'(m) > 0$. Then $[x_1, x_2]$, $[x_3, x_4]$ are disjoint where $x_i = x_F(m_i)$, $i = 1, 2, 3, 4$.

Proof. From the intermediate value theorem, we have $x_F([m_i, m_{i+1}]) = [x_i, x_{i+1}]$, $i = 1, 3$. Suppose that $x_0 \in [x_1, x_2] \cap [x_3, x_4]$ and $x_0 \neq 0$. Then $\exists\, \bar{m}_i \in [m_i, m_{i+1}]$, $i = 1, 3$ such that $x_F(\bar{m}_1) = x_F(\bar{m}_3) = x_0$. For any $\hat{m} \in (m_1, m_2) \cup (m_3, m_4)$, we have from Theorem 4.2 $\hat{x} = x_F(\hat{m}) \in {S_F}^c$ and $\hat{m} = m_F(\hat{x})$. Because of the continuity of $m(z)$ on $\overline{D} - \{0\}$ (Theorem 1.1), we have $m_F(\hat{x}) = \hat{m} \in (m_i, m_{i+1}) \to m_i$ as $\hat{x} \in (x_i, x_{i+1}) \to x_0$, $i = 1, 3$. Thus the limit of $m(z)$ does not exist at x_0, a contradiction.

We can therefore assume that $x_2 = x_3 = 0$. If $0 \in {S_F}^c$, then $m(z)$ is analytic at 0, and therefore $m(z)$ is continuous on $\overline{D}$, and, from the previous argument, we arise at a contradiction.

If $0 \in S_F$, then, since $(x_1, 0) \cup (0, x_4) \subset S_F^c$, F has positive mass, say α, at 0. Therefore $m_F(z) = -\frac{\alpha}{z} + \int_{(-\infty, x_1] \cup [x_4, \infty)} \frac{dF(\lambda)}{\lambda - z}$, so that, by the d.c.t., $\lim_{x \uparrow 0} m_F(x) = \infty$. But we must have $\lim_{x \uparrow 0} m_F(x) = m_2$, a contradiction. □

Thus, on any two disjoint intervals in B for which $x_F(m)$ is increasing for all the points in the interior of those intervals, the images of those intervals under $x_F(m)$ are disjoint.

Theorem 4.5. If $(-\infty, b) \subset B$ $((b, \infty) \subset B)$, then $x_F(m) \to 0$ as $m \to -\infty$ $(m \to \infty)$.

Proof. Follows easily from the monotone convergence theorem. □

5. Behavior Near a Boundary Point. Let $a \in \mathbb{R} - \{0\}$ be a point in ∂S_F. For convenience, throughout this section we will assume a to be a right end-point of an interval of S_F^c, that is, $[a-\delta, a) \in S_F^c$ for some $\delta > 0$. The analysis on the other possibility for a will follow analogously.

Let $m_a = m_0(a)$ (defined in (1.5)). We know from Theorem 1.1 and the previous section that $\lim_{m \uparrow m_a} x_F(m) = a$, and $x_F'(m) > 0$ for $m \in [m_0(a-\delta), m_a)$.

Theorem 5.1. $m_a \neq 0$.

Proof. Suppose not. Then, for some $\epsilon > 0$, $(-\epsilon, 0) \subset B$, which implies $(\epsilon^{-1}, \infty) \subset S_H^c$. Writing $mx_F(m) = -1 + c\int \frac{\lambda m\, dH(\lambda)}{1+\lambda m}$, we see the integral must converge to c^{-1} as $m \uparrow 0$. However, choosing $M > \epsilon^{-1}$, we have by d.c.t. $\int \frac{\lambda m\, dH(\lambda)}{1+\lambda m} = \int_{(-\infty, M]} \frac{\lambda m\, dH(\lambda)}{1+\lambda m} \to 0$ as $m \uparrow 0$, a contradiction. □

Notice that Theorem 5.1 is still true if $a = 0$ provided $\lim_{x \uparrow 0} m_0(x) < \infty$, with m_a defined to be this limit (the only other possibility being $x_F'(m) > 0$ for all m sufficiently large (see Theorem 4.5)).

It is still possible for $-m_a^{-1} \in S_H$, that is, when $H\{[-m_a^{-1}, -m_a^{-1}+\epsilon]\} > 0$ $\forall \epsilon > 0$ and $\lim_{m \uparrow m_a} \int \frac{\lambda^2 dH(\lambda)}{(1+\lambda m)^2}$ exists, so for c sufficiently small, $x_F' > 0$ on $[m_0(a-\delta), m_a)$. This can only occur when H is not discrete, for example, when $H(x) = (x-1)^3$ for $x \in [1,2]$. Further work is needed in this case. Preliminary analysis shows the behavior of f near 0 to deviate from what will be derived below.

We henceforth assume $m_a \in B$. Then, from Theorem 4.3 and the fact that x_F cannot be constant, a local maximum for x_F must occur at m_a, that is, $x_F' < 0$ on an interval to the right of m_a. We assume, again, that $H \neq 1_{[0,\infty)}$.

Theorem 5.2. $x_F''(m_a) < 0$.

Proof. Suppose $x_F''(m_a) = 0$. Then it follows that $\int \frac{\lambda^2}{(1+\lambda m_a)^3} dH(\lambda) = 0$ and $x'''(m_a) = -\frac{6c}{m_a^2} \int \frac{\lambda^2 dH(\lambda)}{(1+\lambda m_a)^4} < 0$. But since the first nonvanishing derivative of a function at a local extreme must be of even order, we arise at a contradiction. □

Let us now establish an expression for f near and to the right of a, which displays its similarity to the square root function. Write $m_0(x) = m_1(x) + im_2(x)$. When $m_2(x) > 0$, $m_0 = m_0(x)$ satisfies (1.6), so that x, $m_j = m_j(x)$, $j = 1, 2$, satisfy (3.1) with $y = 0$. It follows that

$$x = c\int \frac{\lambda\, dH(\lambda)}{(1+\lambda m_1)^2 + \lambda^2 m_2^2} \quad \text{and} \quad 0 = \frac{1}{m_1^2+m_2^2} - c\int \frac{\lambda^2 dH(\lambda)}{(1+\lambda m_1)^2 + \lambda^2 m_2^2}. \tag{5.1}$$

We have $(m_1(a), m_2(a)) = (m_a, 0)$. Choose δ and ϵ sufficiently small so that $0 \notin (a, a+\delta)$, $x \in (a, a+\delta) \implies m_1(x) \in (m_a - \epsilon, m_a + \epsilon) \subset B$, and $x_F(m_a - \epsilon) \in S_F^c$.

We argue that $f(x) = \frac{1}{\pi} m_2(x) > 0$ for all $x \in (a, a+\delta)$. Suppose $x_0 \in (a, a+\delta)$ is such that $m_2(x_0) = 0$. Letting $z \in D \to x_0$ and using (1.4) together with Theorem 1.1, we find, with $m_1 = m_1(x_0)$, $x_0 = x_F(m_1)$, and from the second equation in (3.1), $x_F'(m_1) \geq 0$. Therefore, from Theorem 4.3, $m_1 \in [m_a - \epsilon, m_a]$, forcing $x_0 \leq a$, a contradiction.

Therefore, $f > 0$ on $(a, a+\delta)$, and (5.1) holds for all $x \in (a, a+\delta)$, $m_j = m_j(x)$, $j = 1, 2$. Differentiating implicitly both equations in (5.1) with respect to x, it is straightforward to derive $m_2 m_2' = \frac{m_1 A_2 + (m_1^2 - m_2^2) A_3}{(A_2 + A_3 m_1)^2 + A_3^2 m_2^2}$, where $A_j = 2c \int \frac{\lambda^j dH(\lambda)}{((1+\lambda m_1)^2 + \lambda^2 m_2^2)^2}$, $j = 2, 3$. For $x \in (a, a+\delta)$, let $g(x) = 2m_2(x) m_2'(x)$, and $g(a) = -C_2{}^{-1}$ where $C_2 \equiv \frac{x_F''(m_a)}{2}$. It is straightfoward to verify that g is right continuous at a. Thus for $x \in [a, a+\epsilon)$

$$m_2(x) = \left(\int_a^x g(t) dt \right)^{\frac{1}{2}}. \tag{5.2}$$

We remark that the above argument will carry over for the case $a = 0$, as long as $\lim_{z \in D \to 0} m_F(z)$ exists and is contained in B.

From Theorem 5.2 it follows from an argument given in [2] that for $x > a$,

$$f(x) = \frac{1}{\pi} \left(\frac{a - x}{C_2} \right)^{\frac{1}{2}} (1 + o(1)). \tag{5.3}$$

From L'Hopital's rule, we see that (5.2) gives us (5.3). But (5.2) yields additionally $\lim_{x \downarrow a} m_2'(x) / \sqrt{\frac{a-x}{C_2}}' = 1$, demonstrating $f(x)$ to have more in common with $\frac{1}{\pi} \left(\frac{a-x}{c_2} \right)^{\frac{1}{2}}$ than what can be inferred from (5.3).

6. Conclusion. The results in this paper provide general analytic properties of F, and, consequently, on F_0 via (1.1), for arbitrary H, along with an analysis of x_F sufficient enough to allow the determination of S_F through its graph. The former is particularly relevant in multivariate statistics, where the eigenvalues of a population covariance matrix of sizable dimension need to be inferred from those of a sample covariance matrix resulting from a sample size insufficient to permit the use of conventional estimation methods. One scheme to determine H, the spectral distribution of the population matrix, follows from the way property *i*) in the introduction is verified, namely through Stieltjes transforms and the inverse of m_F. Using the properties of F established in this paper, that is, its analyticity away from the origin, and the "square root" behavior of its derivative near boundary points (which include all non-zero points in ∂S_F, since now H is discrete), an approximation of F (which only manifests itself in the limit), and, consequently, m_F, can be made by an appropriate smoothing of the spectral distribution of the sampled matrix. The approximation would hopefully be an improvement over simply using the sampled eigenvalues. Research along these lines is currently being pursued.

Other fundamental properties which can be proven using (1.3),(1.4) concern information beyond property *ii*) in the introduction when S_H contains boundary points. By tracking the relative extreme values of x_F, it is straightforward to show the (eventual existence and) convergence of points in ∂S_{F_0} to their corresponding points in $(\partial H) - \{0\}$ as $c \to 0$. Moreover, for every $[b_1, b_2] \subset S_H$ for which $0 \notin [b_1, b_2]$, and $(b_1 - \epsilon, b_1) \cup (b_2, b_2 + \epsilon) \subset S_{F_0}^c$ for some $\epsilon > 0$, the corresponding interval $[a_1, a_2] \subset S_{F_0}$ (when it exists for c sufficiently small) satisfies $F_0\{[a_1, a_2]\} = H\{(b_1, b_2]\}$ (see [5] for the case when $S_H \subset [0, \infty)$). These and other results concerning the continuous dependence of F_0 on c will appear in future work.

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