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Comments on a Result of Yin, Bai, and Krishnaiah for Large Dimensional Multivariate F Matrices

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A theorem in Yin, Bai, and Krishnaiah (*J. Multivariate Anal.* **13** (1983), 508–516) shows that the smallest eigenvalue of a class of large dimensional sample covariance matrices stays almost surely bounded away from zero. The theorem assumes a certain restriction on the class of matrices. With slight modifications of the proof in op cit, it is shown here that the theorem is true for all relevant matrices.

Let $\{Y_{ij}\} i, j = 1, 2,\ldots$, be i.i.d. $N(0, 1)$ random variables. For $n = 1, 2,\ldots$, let $Y_n = (Y_{ij})\ i = 1, 2,\ldots, p;\ j = 1, 2,\ldots, n$, where $p/n \to y \in (0, 1)$ as $n \to \infty$, and let $A_n = 1/n\ Y_n Y_n^T$.

Let $\underline{\lambda}_n$ and $\bar{\lambda}_n$ denote, respectively, the smallest and largest eigenvalues of A_n. In proving their result on the limiting eigenvalue distribution of large dimensional multivariate F matrices Yin, Bai, and Krishnaiah [1] needed to show that $\underline{\lambda}_n$ stayed almost surely bounded away from 0 as $n \to \infty$. They were successful in showing this property for $y < \frac{1}{2}$. Their result follows from the following statement of Theorem 3.2 [1]: Let $y < \frac{1}{2}$. Then

$$P(\underline{\lambda}_n \leqslant \varepsilon) \leqslant CD^n \varepsilon^{\alpha n} \qquad 0 < \varepsilon \leqslant \varepsilon_0, \tag{1}$$

where C, D, and α are positive constants.

The result on $\underline{\lambda}_n$ easily follows: (1) implies $\lim_{n\to\infty} \underline{\lambda}_n \geqslant^{a.s.} \varepsilon$ for any $\varepsilon \in (0, \varepsilon_0]$ such that $D\varepsilon^\alpha < 1$.

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With slight modifications in the proof of Theorem 3.2 we will presently show the truth of (1) for *all* $y \in (0, 1)$, extending the result on multivariate F matrices to all possible cases. We will indicate only the changes in the proof, referring the reader to [1] for the remaining details.

Choose $\beta < \frac{1}{4}$ and $\gamma > 0$ such that $\gamma < 2\beta(1-y)$. Then

$$\frac{1}{2} - 2\beta y - \beta(1-y) > \frac{\gamma}{2}. \tag{2}$$

Let $\delta = 2\beta(1-y) - \gamma$. For the moment, let $\varepsilon > 0$ be such that $\varepsilon^{\beta} < \frac{1}{2}$. Let $r = \varepsilon^{\beta}$ and $K = \varepsilon^{-\delta}$.

We can improve the bound on $x_k^T A_n x_k$ given in [1, p.512], namely

$$\begin{aligned} x_k^T A_n x_k &= (x_k - z)^T A_n (x_k - z) + 2z^T A_n x_k - z^T A_n z \\ &\leqslant Kr^2 + 2\,\|x_k\| (z^T A_n^2 z)^{1/2} \leqslant Kr^2 + 2\,\|x_k\|\, K^{1/2} (z^T A_n z)^{1/2} \\ &\leqslant Kr^2 + 4K^{1/2}\varepsilon^{1/2}. \end{aligned}$$

With this bound positive constants C_1, D_1, and $\underline{\varepsilon}_0$ exist for which

$$P(\underline{\lambda}_n \leqslant \varepsilon) \leqslant C_1 \left(((2\pi e)^{p/2n}\, e^{1/2})^n \cdot \left(\frac{Kr^2 + 4K^{1/2}e^{1/2}}{r^{2p/n}} \right)^{n/2} + e^{-D_1 K n} \right)$$

for all $\varepsilon \in (0, \underline{\varepsilon}_0]$. We have

$$\frac{Kr^2 + 4K^{1/2}\varepsilon^{1/2}}{r^{2p/n}} = \varepsilon^{\gamma + 2\beta(y - p/n)} + 4\varepsilon^{\gamma/2 + 1/2 - (2p/n)\beta - \beta(1-y)}.$$

Since $p/n \to y$ as $n \to \infty$ we have from (2) $\gamma/2 + 1/2 - 2\beta(p/n) - \beta(1-y) > \gamma$ for all n sufficiently large.

Choose $\alpha > 0$ such that $\alpha < \gamma/2$. Then we can find a $D > 0$ independent of ε such that for every $\varepsilon \in (0, \underline{\varepsilon}_0]$ and for all n sufficiently large (independent of ε)

$$P(\underline{\lambda}_n \leqslant \varepsilon) \leqslant C_1 (D^n \varepsilon^{\alpha n} + e^{-D_1 \varepsilon^{-\delta} n}).$$

Let $\varepsilon_0 > 0$ be such that $\varepsilon_0 \leqslant \underline{\varepsilon}_0$ and $D\varepsilon^{\alpha} e^{D_1 \varepsilon^{-\delta}} \geqslant 1$ for all $\varepsilon \in (0, \varepsilon_0]$. Then for all n sufficiently large and for all $\varepsilon \in (0, \varepsilon_0]$ we have

$$P(\underline{\lambda}_n \leqslant \varepsilon) \leqslant 2C_1 D^n \varepsilon^{\alpha n}.$$

Finally, choose a $C > 0$ for which (1) holds for all n.

REFERENCES

[1] YIN, Y. Q., BAI, Z. D., AND KRISHNAIAH, P. R. (1983). Limiting behavior of the eigenvalues of a multivariate F matrix. *J. Multivariate. Anal.* **13**, 508–516.