Homogeneous and isotropic random field $X(t)$, $t \in \mathbb{R}^2$, on the plane admits the spectral representation

$$X(t,\varphi) = \sum_{m=0}^{\infty} \cos(m\varphi) Y_m(r\lambda) Z_m^1(d\lambda) + \sum_{m=1}^{\infty} \sin(m\varphi) Y_m(r\lambda) Z_m^2(d\lambda).$$

These spectral decompositions of random fields form a power tool for the solution of statistical problems for random fields such as extrapolation, interpolation, filtering, and estimation of parameters of the distribution (Yadrenko 1983; Yaglom 1987a, b).

About the Author

Dr. Mikhail P. Moklyachuk is a Professor of the Department of Probability Theory, Statistics and Actuarial Mathematics, Kyiv National Taras Shevchenko University, Ukraine. He is the author and coauthor of more than 100 papers and six books, including *Robust estimates for functionals of stochastic processes* (Kyiv University Press, 2008). Professor Moklyachuk has received the Taras Shevchenko prize (Kyiv University best textbook award, 1999) for the textbook *Variational Calculus. Extremum Problems*. He is the editor of the Cooperation Unit of Zentralblatt MATH (Zentralblatt fuer Mathematik/Mathematics Abstracts), coeditor of *Current Index to Statistics*, and member of the editorial board, *Theory of Probability and Mathematical Statistics*.

Cross References

►Estimation Problems for Random Fields
►Measure Theory in Probability
►Model-Based Geostatistics
►Random Variable
►Spatial Statistics
►Stochastic Processes

References and Further Reading

Chung KL, Walsh JB (2005) Markov processes, Brownian motion, and time symmetry, 2nd ed. Springer, New York, NY
Glimm J, Jaffe A (1981) Quantum physics: a functional integral point of view. Springer, Berlin/Heidelberg/New York
Kerstan J, Matthes K, Mecke J (1974) Mathematische Lehrbücher und Monographien. II. Abt. Mathematische Monographien. Band XXVII. Akademie, Berlin
Malyshev VA, Minlos RA (1985) Stochastic Gibbs fields. The method of cluster expansions. Nauka, Moskva
Monin AS, Yaglom AM (2007a) Statistical fluid mechanics: mechanics of turbulence, volume I. Edited and with a preface by Lumley JL, Dover, Mineola, NY
Monin AS, Yaglom AM (2007b) Statistical fluid mechanics: mechanics of turbulence, volume II. Edited and with a preface by Lumley JL, Dover, Mineola, NY
Rozanov YuA (1982) Markov random fields. Springer, New York
Yadrenko MI (1983) Spectral theory of random fields. Translation Series in Mathematics and Engineering. Optimization Software, Publications Division, New York; Springer, New York
Yaglom AM (1987a) Correlation theory of stationary and related random functions. volume I. Basic results. Springer Series in Statistics. Springer, New York
Yaglom AM (1987b) Correlation theory of stationary and related random functions, volume II. Supplementary notes and references. Springer Series in Statistics. Springer, New York

Random Matrix Theory

Jack W. Silverstein
Professor
North Carolina State University, Raleigh, NC, USA

Random matrix theory (RMT) originated from the investigation of energy levels of a large number of particles in quantum mechanics. Many laws were discovered by numerical study in mathematical physics. In the late 1950s, E. P. Wigner formulated the problem in terms of the empirical distribution of a random matrix (Wigner 1955, 1958), which began the investigation into the semicircular law of Gaussian matrices. Since then, RMT has formed an active branch in modern probability theory.

Basic Concepts

Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. If all λ_js are real, then we can construct a 1-dimensional empirical distribution function

$$F^{\mathbf{A}}(x) = \frac{1}{n} \sum_{j=1}^{n} I(\lambda_j \le x),$$

otherwise, we may construct a 2-dimensional empirical distribution function by the real and imaginary parts of λ_j, i.e.

$$F^{\mathbf{A}}(x,y) = \frac{1}{n} \sum_{j=1}^{n} I(\Re(\lambda_j) \le x; \Im(\lambda_j) \le y).$$

Then, $F^{\mathbf{A}}$ is called the *empirical spectral distribution* (ESD) of $\mathbf{A}$. The main task of RMT is to investigate limiting properties of $F^{\mathbf{A}}$ in the case where $\mathbf{A}$ is random and the order n tends to infinity. If there is a limit distribution F, then the limit is called the *limiting spectral distribution* (LSD) of the sequence of the $\mathbf{A}$. Interesting problems include finding the explicit forms of the LSD if it exists and to investigate its properties.

There are two methods used in determining limiting properties of $F^{\mathbf{A}}$ (Bai 1999). One is the *method of moments*, using the fact that the moments of $F^{\mathbf{A}}$ are the scaled traces of powers of **A**. The other is using *Stieltjes transforms*, defined for any distribution function F as

$$m(z) = \int \frac{1}{x-z} dF(x),$$

for $z \in \mathbb{C}$.

Contrary to the progress made on the eigenvalues of large dimensional random matrices, very few results have been obtained on the limiting properties of the eigenmatrix (i.e., the matrix of the standardized eigenvectors of **A**). Due to its importance in the application to statistics and applied areas, investigation on eigenmatrices is becoming more active.

Limiting Spectral Distributions

1. **Semicircular Law** A Wigner matrix is defined as a Hermitian (symmetric if real) matrix $\mathbf{W} = (w_{ij})_{n\times n}$ whose entries above or on the diagonal are independent. Then the ESD of $n^{-1/2}\mathbf{W}$ tends to the semicircular law with density

$$p(x) = \frac{1}{2\pi}\sqrt{4-x^2}I(|x|<2),$$

if $\mathrm{E}w_{ij} = 0$, $\mathrm{E}|w_{ij}|^2 = 1$ and for any $\delta > 0$,

$$\frac{1}{n^2}\sum_{ij} \mathrm{E}\left|w_{ij}^2\right| I\left(|w_{ij}| \geq \delta\sqrt{n}\right) \to 0.$$

2. **Marcĕnko–Pastur Law** Let $\mathbf{X} = (x_{ij})_{p\times n}$ whose entries are independent random variables with mean zero and variance 1. If $p/n \to y \in (0,\infty)$ and for any $\delta > 0$,

$$\frac{1}{np}\sum_{ij} \mathrm{E}\left|x_{ij}^2\right| I\left(|x_{ij}| \geq \delta\sqrt{n}\right) \to 0.$$

Then the ESD of $\mathbf{S}_n = \frac{1}{n}\mathbf{X}\mathbf{X}^*$ (so-called sample covariance matrix) tends to the Marcĕnko–Pastur law with density

$$\frac{1}{2\pi xy}\sqrt{(b-x)(x-a)}I(a<x<b)$$

where $a = (1-\sqrt{y})^2$ and $b = (1+\sqrt{y})^2$. Furthermore, if $y > 1$, the LSD has a point mass $1-1/y$ at the origin.

3. **LSD of Products of Random Matrices** Let $\mathbf{T}$ $(p\times p)$ be a Hermitian matrix with LSD H (a probability distribution function) and $\mathbf{S}_n$, p/n satisfy the conditions in item (2). Then the ESD of $\mathbf{S}_n\mathbf{T}$ exists and the Stieltjes transform $m(z)$ is the unique solution on the upper complex plane to the equation

$$m = \int \frac{1}{t(1-y-yzm)-z} dH(t),$$

where z is complex with positive imaginary part.

Extreme Eigenvalues and Spectrum Separation

Limits of extreme eigenvalues of large random matrices is one of the important topics. In many cases, under the assumption of finite fourth moment, the extreme eigenvalues almost surely tend to the respective boundaries of the LSD. For the product $\mathbf{S}_n\mathbf{T}$, if the support of the LSD is disconnected, then, under certain conditions, it is proved that there are no eigenvalues among the gaps and the numbers on each side are exactly the same of eigenvalues of **T**, on the corresponding sides of the interval which determines the gap of the LSD (Bai and Silverstein 1999).

Further deeper investigation into extreme eigenvalues is the Tracy–Widom Law which says that $n^{2/3}$ times the difference of the extreme eigenvalues and the corresponding boundary points tends to the so-called Tracy–Widom law (Tracy and Widom 1994).

Convergence Rates of Empirical Spectral Distributions

Convergence rates of ESDs of large dimensional random matrices to their corresponding LSDs are important for application of spectral theory of large dimensional matrix. Bai inequality is the basic mathematical tool to establish the convergence rates (Bai 1993a,b). The currently known best rates are that $O(n^{-1/2})$ for the expected ESDs for Wigner matrix and for sample covariance matrix, and $O_p(n^{-2/5})$ and $O_{a.s.}(n^{-2/5+\eta})$ for their ESDs.

The exact rates are still far from known.

CLT of LSS

If $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of the random matrix **A** and f is a function defined on the space of the eigenvalues, then the LSS (linear spectral statistic) for the random matrix is defined by

$$\frac{1}{n}\sum_{k=1}^{n} f(\lambda_k) = \int f(x)dF^{\mathbf{A}}(x).$$

To investigate the limiting distribution of the LSS, we define $X_n(f) = n\left(\int f(x)d\left(F^{\mathbf{A}}(x) - F(x)\right)\right)$.

Under certain conditions, the normalized LSS, $X_n(f)$, is proved to tend to a normal distribution for the Wigner matrix, the product $\mathbf{S}_n\mathbf{T}$, as well as for the multivariate F-matrix, with asymptotic means and variances explicitly expressed by the Stieltjes transforms of the LSDs (Bai and Yao 2005; Bai and Silverstein 2004; Zheng 2010).

These theorems have been found to have important applications to multivariate analysis and many other areas.

Limiting Properties of Eigenvectors

Work in this area has been primarily done on the matrices in item (2) with **X** containing real entries (Silverstein

1979, 1984, 1990). Write $\mathbf{S}_n = \mathbf{O}\mathbf{\Lambda}\mathbf{O}^*$, its spectral decomposition. When the entries of $\mathbf{X}$ are Gaussian, then $\mathbf{S}_n$ is the standard Wishart matrix, with $\mathbf{O}$ Haar-distributed in the group of $p \times p$ orthogonal matrices. The question is to compare the distribution of $\mathbf{O}$ when the entries of $\mathbf{X}$ are not Gaussian to Haar measure when p is large. This has been pursued when $\mathbf{X}$ is made up of iid random variables, by comparing the distribution of $\mathbf{y} = \mathbf{O}^*\mathbf{x}$, where $\mathbf{x}$ is a unit p-dimensional vector, to the uniform distribution on the unit sphere in $\mathbb{R}^p$. A stochastic process is defined in terms of the entries of $\mathbf{y}$, which converges weakly to Brownian bridge in the Wishart case. A necessary condition for this process to behave the same way for non Gaussian entries has been shown to be $\mathrm{E}\left(x_{11}^4\right) = 3$, matching the fourth moment of a standardized Gaussian (Silverstein 1984). For certain choices of $\mathbf{x}$ and for symmetrically distributed x_{11}, weak convergence to Brownian bridge has been shown in Silverstein (1990).

About the Author

Professor Silverstein was named IMS Fellow for "seminal contributions to the theory and application of random matrices" (2007). He has (co-)authored over 50 publications, including the book *Spectral Analysis of Large Dimensional Random Matrices* (with Z.D. Bai, 2nd edition, Springer, New York, 2009).

Cross References

▸Eigenvalue, Eigenvector and Eigenspace
▸Ergodic Theorem
▸Limit Theorems of Probability Theory
▸Multivariate Statistical Distributions
▸Statistical Inference for Quantum Systems

References and Further Reading

Bai ZD (1999) Methodologies in spectral analysis of large dimensional random matrices: a review. Stat Sinica 9(3):611–677

Bai ZD (1993a) Convergence rate of expected spectral distributions of large random matrices. Part I. Wigner matrices. Ann Probab 21(2):625–648

Bai ZD (1993b) Convergence rate of expected spectral distributions of large random matrices. Part II. Sample covariance matrices. Ann Probab 21(2):649–672

Bai ZD, Silverstein JW (2004) CLT for linear spectral statistics of large-dimensional sample covariance matrices. Ann Probab 32(1):553–605

Bai ZD, Silverstein JW (1999) Exact separation of eigenvalues of large dimensional sample covariance matrices. Ann Probab 27(3):1536–1555

Bai ZD, Yao JF (2005) On the convergence of the spectral empirical process of Wigner matrices. Bernoulli 11(6):1059–1092

Silverstein JW (1979) On the randomness of eigenvectors generated from networks with random topologies. SIAM J Appl Math 37:235–245

Silverstein JW (1984) Some limit theorems on the eigenvectors of large dimensional sample covariance matrices. J Multivariate Anal 15(3):295–324

Silverstein JW (1990) Weak convergence of random functions defined by the eigenvectors of sample covariance matrices. Ann Probab 18:1174–1194

Tracy CA, Widom H (1994) Level-spacing distributions and the Airy kernel. Commun Math Phys 159:151–174

Wigner EP (1955) Characteristic vectors bordered matrices with infinite dimensions. Ann Math 62:548–564

Wigner EP (1958) On the distributions of the roots of certain symmetric matrices. Ann Math 67:325–327

Zheng S (2010) Central limit theorem for linear spectral statistics of large dimensional F-Matrix, to appear in Ann Inst Henri Poincaré Probab Stat

Random Permutations and Partition Models

Peter McCullagh
John D. MacArthur Distinguished Service Professor
University of Chicago, Chicago, IL, USA

Set Partitions

For $n \geq 1$, a partition B of the finite set $[n] = \{1, \ldots, n\}$ is

- A collection $B = \{b_1, \ldots\}$ of disjoint non-empty subsets, called blocks, whose union is $[n]$
- An equivalence relation or Boolean function $B\colon [n] \times [n] \to \{0, 1\}$ that is reflexive, symmetric and transitive
- A symmetric Boolean matrix such that $B_{ij} = 1$ if i, j belong to the same block

These equivalent representations are not distinguished in the notation, so B is a set of subsets, a matrix, a Boolean function, or a subset of $[n]\times[n]$, as the context demands. In practice, a partition is sometimes written in an abbreviated form, such as $B = 2|13$ for a partition of $[3]$. In this notation, the five partitions of $[3]$ are

$$123, \quad 12|3, \quad 13|2, \quad 23|1, \quad 1|2|3.$$

The blocks are unordered, so $2|13$ is the same partition as $13|2$ and $2|31$.

A partition B is a sub-partition of B^* if each block of B is a subset of some block of B^* or, equivalently, if $B_{ij} = 1$ implies $B^*_{ij} = 1$. This relationship is a partial order denoted by $B \leq B^*$, which can be interpreted as $B \subset B^*$ if each partition is regarded as a subset of $[n]^2$. The partition lattice $\mathcal{E}_n$ is the set of partitions of $[n]$ with this partial order. To each pair of partitions B, B' there corresponds a greatest lower bound $B \wedge B'$, which is the set intersection or Hadamard component-wise matrix product. The least upper bound