LIMITING EIGENVALUE BEHAVIOR OF A CLASS OF LARGE DIMENSIONAL RANDOM MATRICES FORMED FROM A HADAMARD PRODUCT

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Abstract. This paper investigates the strong limiting behavior of the eigenvalues of the class of matrices $\frac{1}{N}(D_n \circ X_n)(D_n \circ X_n)^*$, studied in Girko 2001. Here, $X_n = (x_{ij})$ is an $n \times N$ random matrix consisting of independent complex standardized random variables, $D_n = (d_{ij})$, $n \times N$, has nonnegative entries, and $\circ$ denotes Hadamard (componentwise) product. Results are obtained under assumptions on the entries of X_n and D_n which are different from those in Girko (2001), which include a Lindeberg condition on the entries of $D_n \circ X_n$, as well as a bound on the average of the rows and columns of $D_n \circ D_n$. The present paper separates the assumptions needed on X_n and D_n. It assumes a Lindeberg condition on the entries of X_n, along with a tigntness-like condition on the entries of D_n,

1. Introduction

This paper deals with the limiting eigenvalue behavior of the class of Hermitian nonnegative definite matrices

$$B_n = \frac{1}{N}(D_n \circ X_n)(D_n \circ X_n)^*, \tag{1.1}$$

where for each positive integer n $X_n = (x_{ij}^{(n)})$ is $n \times N$ with random variables $x_{ij}^{(n)} \in \mathbb{C}$, independent, and standardized ($\mathsf{E}x_{ij}^{(n)} = 0$, $\mathsf{E}|x_{ij}^{(n)}|^2 = 1$), D_n is $n \times N$ containing nonrandom, nonnegative real numbers $d_{ij} = d_{ij}^n$, $\circ$ denotes Hadamard product, and $N = N(n)$ with $0 < \liminf_n n/N \leq \limsup_n n/N < \infty$. Such matrices arise in various situations when the dimension is large and where there is no prescribed structure to the elements in the D_n matrix.

A standard way to pursue the limiting eigenvalue behavior of Hermitian random matrices as the dimension increases is through the empirical distribution function (e.d.f) of their eigenvalues, that is, for random Hermitian $n \times n$ matrix A_n, let for every $x \in \mathbb{R}$, $F^{A_n}(x) = \{\text{number of eigenvalues of } A_n \leq x\}/n$. A standard tool used in understanding the (e.d.f.) of the eigenvalues has been since Marčenko Pastur (1967), the Stieltjes transform, where for arbitrary finite measure μ on $\mathbb{R}$ is defined by

$$m_\mu(z) = \int \frac{1}{x-z} d\mu(x), \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \Im > 0\}. \tag{1.2}$$

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It is analytic and takes values in $\mathbb{C}^+$. Notice that $|m_\mu(z)| \le \mu(\mathbb{R})/\Im z$. The Stieltjes transform of the measure induced by F^{A_n} is then

$$m_{A_n} = \int \frac{1}{x-z} dF^{A_n}(x) = \frac{1}{n}\mathsf{tr}\,(A_n - zI)^{-1},$$

where I is the $n \times n$ identity matrix. and tr is the trace of a matrix.

Due to the inversion formula

$$\mu([a,b]) = \frac{1}{\pi} \lim_{\eta\to 0^+} \int_a^b \Im m_\mu(\xi + i\eta) d\xi \quad (a,\, b \text{ continuity points of } \mu), \tag{1.3}$$

it will follow that understanding the limiting behavior of F^{A_n} can be handled by its Stieltjes transform.

Work on the eigenvalue behavior of B_n has been done in [Girko 2001]. Indeed, Theorem 10.1 of [Girko 2001], contains the following result: assume $x_{ij}^{(n)} \in \mathbb{R}$,

$$\sup_n \max_{\substack{i=1,\dots,n\\ j=1,\dots,N}} \left\{ \frac{1}{n}\sum_{i=1}^n d_{ij}^2 + \frac{1}{N}\sum_{j=1}^N d_{ij}^2 \right\} < \infty, \tag{1.4}$$

and a Lindeberg condition is satisfied, namely for arbitrary $\eta > 0$

$$\lim_{n\to\infty} \max_{\substack{i=1,\dots,n\\ j=1,\dots,N}} \left\{ \frac{1}{n}\sum_{i=1}^n d_{ij}^2 \mathsf{E}((x_{ij}^{(n)})^2 I(d_{ij}|x_{ij}^{(n)}| > \eta\sqrt{n})) \right.$$
$$\left. + \frac{1}{N}\sum_{j=1}^N d_{ij}^2 \mathsf{E}((x_{ij}^{(n)})^2 I(d_{ij}|x_{ij}^{(n)}| > \eta\sqrt{n})) \right\} = 0,$$

where $I(A)$ is the indicator function on the set A. Then, with $\|\cdot\|$ denoting the sup norm on functions, for each n, there exists a nonrandom probability distribution function F_n^0, such that, with probability one

$$\lim_{n\to\infty} \|F^{B_n} - F_n^0\| = 0.$$

and F_n^0 has Stieltjes transform

$$G_n(z) := \frac{1}{n}\sum_{i=1}^n \frac{1}{\frac{1}{N}\sum_{k=1}^N \frac{d_{ik}^2}{1+\frac{n}{N}e_k^0(z)} - z}, \tag{1.5}$$

where for each $z \in \mathbb{C}^+$, $e_1^0, \dots, e_N^0$ are unique solutions lying in $\mathbb{C}^+$ to the system of equations

$$e_j^0(z) = \frac{1}{n}\sum_{i=1}^n \frac{d_{ij}^2}{\frac{1}{N}\sum_{k=1}^N \frac{d_{ik}^2}{1+\frac{n}{N}e_k^0(z)} - z}. \tag{1.6}$$

This result is among a collection which differs from typical results on limiting eigenvalue behavior (as the dimension of the matrix increases), in that there is no statement on what a possible limiting e.d.f of the eigenvalues of B_n could be. However, the result is important in that it shows that F^{B_n} is becoming less random, with a way for deriving, through (1.3), what it is close to. The F_n^0's can be thought of as *deterministic equivalents* of the e.d.f.'s

The aim of the current paper is to prove a result under different assumptions, in particular, for each $\eta > 0$

$$\lim_{n\to\infty} \frac{1}{nN} \sum_{jk} \mathsf{E}|x_{jk}^{(n)}|^2 I(|x_{jk}^{(n)}| \geq \eta\sqrt{n}) = 0, \tag{1.7}$$

From this it is straightforward to construct a sequence $\{\eta_n\}$ of positive numbers for which $\eta_n \downarrow 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{1}{\eta_n^2 nN} \sum_{jk} \mathsf{E}|x_{jk}^{(n)}|^2 I(|x_{jk}^{(n)}| \geq \eta_n\sqrt{n}) = 0. \tag{1.8}$$

We also assume the existence of positive e and f such that for all n sufficiently large

$$e < \eta_n\sqrt{n} \quad \text{and} \quad \mathsf{E}\left|x_{jk}^{(n)} I(|x_{jk}^{(n)}| \leq e) - \mathsf{E}(x_{jk}^{(n)} I(|x_{jk}^{(n)}| \leq e))\right|^2 > f \tag{1.9}$$

Also, we assume each column of D_n is nonzero, and the matrix satisfies the following property. For every $\epsilon > 0$ there exists an $M_\epsilon > 0$ such that for each n there exists sets $E_{r\epsilon}^n \subset \{1, 2, \ldots, n\}$, $E_{c\epsilon}^n \subset \{1, 2, \ldots, N\}$ such that

(1) $\#E_{r\epsilon}^n + \#E_{c\epsilon}^n \leq \epsilon n$ (# denotes number of elements in the set)
(2) $d_{jk}^n \leq M_\epsilon$ for $j \in E_{r\epsilon}^{n\ c}$ and $k \in E_{c\epsilon}^{n\ c}$.

The result is expressed in terms of the following metric on sub-probability measures on $\mathbb{R}$ via their distribution functions:

$$D(F, G) \equiv \sum_{i=1}^{\infty} \left| \int f_i dF - \int f_i dG \right| 2^{-i},$$

where $\{f_i\}$ is an enumeration of all continuous functions that take a constant $\frac{1}{m}$ value (m a positive integer) on $[a, b]$ where a, b are rational, 0 on $(-\infty, a - \frac{1}{m}] \cup [b + \frac{1}{m}, \infty)$, and linear on each of $[a - \frac{1}{m}, a], [b, b + \frac{1}{m}]$. It is straightforward to verify that $D(\cdot, \cdot)$ induces the topology of weak convergence on probability measures, and the topology of vague convergence on the set of sub-probability measures. Since for $x, y \in \mathbb{R}$, $|f_i(x) - f_i(y)| \leq |x - y|$, we have for two empirical distribution functions (e.d.f.) F, G on the respective sets $\{x_1, \ldots, x_n)\}, \{y_1, \ldots, y_n\}$

$$D(F, G) \leq \frac{1}{n} \sum_{j=1}^{n} |x_j - y_j| \leq \left(\frac{1}{n} \sum_{j=1}^{n} (x_j - y_j)^2 \right)^{1/2}. \tag{1.10}$$

We will prove

Theorem 1.1. *For each $\epsilon > 0$ choose $d_\epsilon \geq M_\epsilon$ and define $\widetilde{d}_{jk}^n = \widetilde{d}_{jk}^{n\epsilon} = d_{jk}^n I(d_{jk}^n \leq d_\epsilon)$. Then with probability one*

$$\limsup_n D(F^{B_n}, F_{n,\epsilon}^0) \leq \epsilon, \tag{1.11}$$

where $F_{n,\epsilon}^0$ is the probability distribution function having Stieltjes transform

$$G_n(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\frac{1}{N} \sum_{k=1}^{N} \frac{\widetilde{d}_{ik}^{n\ 2}}{1 + \frac{n}{N} e_k^0(z)} - z}, \tag{1.12}$$

and for each $z \in \mathbb{C}^+$, $e_k^0 = e_{n,k}^0(z)$ *are unique solutions lying in* $\mathbb{C}^+$ *to the system of equations*

$$e_j^0(z) = \frac{1}{n}\sum_{i=1}^{n} \frac{\widetilde{d}_{ij}^{n\,2}}{\frac{1}{N}\sum_{k=1}^{N}\frac{\widetilde{d}_{ik}^{n\,2}}{1+\frac{n}{N}e_k^0(z)} - z}. \tag{1.13}$$

Corollary 1.2. *With probability one, there exists a (random) sequence* $\{\epsilon_n\}$, *such that, with* $\widetilde{d}_{ij}^n$ *and* F_{n,ϵ_n}^0 *defined as above, we have*

$$\lim_{n\to\infty} D(F^{B_n}, F_{n,\epsilon_n}^0) = 0. \tag{1.14}$$

The present assumptions allow for a clearer understanding as to the conditions needed on the entries of X_n and D_n separately, so for example, determining the applicability of the theorem on matrix ensembles with the same X_n satisfying (1.7),(1.8),(1.9) would only require investigating the properties on different D_n's. The Lindeberg condition is just assumed on the $x_{ij}^{(n)}$'s, while a tightness-like condition on the d_{ij}'s is only needed. There can be some exceedingly large values of d_{ij}, just not too many of them. The value on the left side of (1.4) can go unbounded.

The conclusion of the Theorem should not be considered substantially weaker than the one in Theorem 10.1 of [Girko 2001]. Typical of eigenvalue results of this nature, it is not known how large the dimension should be in order to obtain prescribed accuracy and reliability. One of the main uses of these limit laws is to be able to identify and understand the underlying assumptions on the makeup of the matrix, typically by viewing a histogram of the random eigenvalues.

The proofs of the Theorem and corollary are given in section 3, following a section on lemmas needed in the proof. Section 3 ends with a scheme to compute e^0.

2. Lemmas

This section contains results on matrix theory and probability need in the proofs.

Lemma 2.1 ([Horn and Johnson, 1990, Corollary 7.3.8])**.** *For* $r\times s$ *matrices* A *and* B *with respective singular values* $\sigma_1 \geq \cdots \geq \sigma_q$, $\tau_1 \geq \cdots \geq \tau_q$ *where* $q = \min\{r,s\}$, *we have*

$$\left(\sum_{i=1}^{q}(\sigma_i - \tau_i)^2\right)^{1/2} \leq \|A-B\|_2,$$

where $\|\cdot\|_2$ *is the Frobenius matrix norm.*

Let $\|\cdot\|$ denote the sup-norm on bounded functions from $\mathbb{R}$ to $\mathbb{R}$. It is clear that for probability distribution functions F and G

$$D(F,G) \leq \|F-G\|.$$

Lemma 2.2. *For matrices* A, B *of the same dimension,* $rank(A+B) \leq rank(A) + rank(B)$.

Lemma 2.3. *Let* A *be* $r\times s$. *Then* $rank(A) \leq$ *number of nonzero rows of* A + *the number of nonzero columns of* A.

Proof. Let $e_m^i \in \mathbb{R}^m$ be the canonical vector with 1 in the $i-th$ position and zero in the remaining positions, $a_{j\cdot}$ the $j-th$ row of A, and $a_{\cdot k}$ the $k-th$ column of A. Then we can write

$$A = \sum_{j=1}^{r} e_r^j a_{j\cdot} = \sum_{k=1}^{s} a_{\cdot k} e_s^{kT} = \frac{1}{2}\sum_{j=1}^{r} e_r^j a_{j\cdot} + \frac{1}{2}\sum_{k=1}^{s} a_{\cdot k} e_s^{kT}.$$

Each of the terms in the sums is a *rank* 1 matrix. Removing all the zero rows and columns and using Lemma 2.2 we get our result. □

For Hermitian matrix A we let F^A denote the e.d.f. of the eigenvalues of A, and for rectangular matrix B we let F^B_{sing} denote the e.d.f. of the singular values of B.

Lemma 2.4 ([Bai and Silverstein, 2010, Theorem A.44])**.** *For $r \times s$ matrices A and B*

$$\|F^{AA^*} - F^{BB^*}\| \le \frac{1}{r} rank(A - B).$$

Since the rank of a matrix is bounded above by the number of its non-zero rows, we have

Lemma 2.5. *For matrices in Lemma 2.4*

$$\|F^{AA^*} - F^{BB^*}\| \le \frac{1}{r}\{\textit{number of non-zero rows of } A - B\}$$
$$\le \frac{1}{r}\{\textit{number of non-zero entries of } A - B\}.$$

Lemma 2.6 (Bernstein's inequality)**.** *Let $X_1, \dots, X_n$ denote independent mean zero random variables uniformly bounded in absolute value by b, $S_n = X_1 + \ldots + X_n$ and $\sigma_n^2 = \mathsf{E} S_n^2$. Then for any $\epsilon > 0$*

$$\mathsf{P}(S_n \ge \epsilon) \le \exp\left(\frac{-\epsilon^2}{2(\sigma_n^2 + \frac{b\epsilon}{3})}\right).$$

Lemma 2.7 (consequence of Burkholder's inequality)**.** *For $\{X_k\}$ independent mean-zero random variables we have for any $p \ge 2$*

$$\mathsf{E}\big|\sum_k X_k\big|^p \le C_p\big(\sum_k \mathsf{E}|X_k|^p + \big(\sum_k \mathsf{E}|X_k|^2\big)^{p/2}\big).$$

Lemma 2.8 ([Bai and Silverstein, 1998])**.** *Let $\{X_k\}$ be a complex martingale difference sequence. Then, for $p > 1$*

$$\mathsf{E}\left|\sum X_k\right|^p \le C_p \mathsf{E}\left(\sum |X_k|^2\right)^{p/2}.$$

Lemma 2.9. *Let B, C be $n \times n$ with B Hermitian, $x \in \mathbb{C}^n$ and $z = x + iv$ with $v > 0$. Then*

$$\frac{1}{|z(1 + x^*(B - zI)^{-1}x)|} \le \frac{1}{v} \quad \textit{and} \quad \frac{1}{|z(1 + \operatorname{tr} C^*(B - zI)^{-1}C)|} \le \frac{1}{v}.$$

Proof. Follows from the fact that the imaginary parts of $x^*((1/z)B - I)^{-1}x$ and $\operatorname{tr} C^*((1/z)B - I)^{-1}C$ are both non-negative. □

Lemma 2.10 ([Bai and Silverstein, 1998]). *Let $z \in \mathbb{C}$ with $v = \Im z > 0$, A, B $n \times n$ with B Hermitian, and $r \in \mathbb{C}^n$. Then*

$$|\mathrm{tr}\,((B - zI)^{-1} - (B + rr^* - zI)^{-1})A| = \left| \frac{r^*(B - zI)^{-1}A(B - zI)^{-1}r}{1 + r^*(B - zI)^{-1}r} \right| \le \frac{\|\!|A|\!\|_2}{v},$$

where $\|\!|\cdot|\!\|_2$ denotes spectral norm.

Lemma 2.11. *For A, r as in Lemma 2.10 with A and $A + rr^*$ both invertible, we have*

$$r^*(A + rr^*)^{-1} = \frac{1}{1 + r^*A^{-1}r} r^*A^{-1}.$$

Proof. *Follows from $r^*A^{-1}(A + rr^*) = (1 + r^*A^{-1}r)r^*$.*

Lemma 2.12 ([Bai and Silverstein, 2010, Lemma B.26]). *Let A be $n \times n$ and $x = (x_1, \dots, x_n)^T$ where the x_i are independent random variables with $\mathsf{E}x_i = 0$, $\mathsf{E}|x_i|^2 = 1$, and $\mathsf{E}|x_i|^\ell \le \nu_\ell$. Then for any $p \ge 1$*

$$\mathsf{E}|x^*Ax - \mathrm{tr}\,A|^p \le C_p \left((\nu_4 \mathrm{tr}\,(AA^*))^{p/2} + \nu_{2p} \mathrm{tr}\,(AA^*)^{p/2} \right).$$

Lemma 2.13 ([Horn and Johnson, 1990, Theorem 8.3.1]). *Let $\rho(C)$ denote the spectral radius of the $N \times N$ matrix C (the largest of the absolute values of the eigenvalues of C). If C contains only nonnegative entries, then $\rho(C)$ is an eigenvalue of C having an eigenvector with nonnegative entries.*

Lemma 2.14 ([Horn and Johnson, 1990, Theorem 8.1.18]). *Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are $N \times N$ with b_{ij} nonnegative and $|a_{ij}| \le b_{ij}$. Then*

$$\rho(A) \le \rho((|a_{ij}|)) \le \rho(B).$$

Lemma 2.15 ([Horn and Johnson, 1991, Lemma 5.7.9]). *Let $A = (a_{ij})$ and $B = (b_{ij})$ be $N \times N$ with a_{ij}, b_{ij} nonnegative. Then*

$$\rho((a_{ij}^{\frac{1}{2}} b_{ij}^{\frac{1}{2}})) \le (\rho(A))^{\frac{1}{2}} (\rho(B))^{\frac{1}{2}}.$$

Lemma 2.16 ([Horn and Johnson, 1990, Lemma 5.6.10]). *For square A and $\epsilon > 0$ there exists a matrix norm $\|\!|\cdot|\!\|$ such that $\rho(A) \le \|\!|A|\!\| \le \rho(A) + \epsilon$.*

Lemma 2.17 ([Horn and Johnson, 1990, Theorem 5.6.26]). *Let $\|\!|\cdot|\!\|$ be a given matrix norm on $n \times n$ matrices. Then there exists an induced matrix norm $N(\cdot)$ on $n \times n$ matrices such that, for any $n \times n$ matrix A we have $N(A) \le \|\!|A|\!\|$.*

From the last two lemmas, we have

Lemma 2.18. *Let A be $n \times n$ satisfy $\rho(A) < 1$. Then there exists a vector norm on $\mathbb{C}^n$, such that with $\|\!|\cdot|\!\|$ denoting its induced matrix norm, we have $\|\!|A|\!\| < 1$.*

Lemma 2.19. *Let A be an $m \times n$ matrix. Then $\|\!|A|\!\|_2 = \|\!|A diag(\omega_1, \dots, \omega_n)|\!\|_2$, where the ω_i are numbers on the unit circle in the complex plane.*

Proof. *Let $y_i \in \mathbb{C}^n$, $i = 1, 2$ be unit vectors for which $\|\!|A|\!\| = \|Ay_1\|_2$ and $\|\!|A diag(\omega_1, \dots, \omega_n)|\!\|_2 = \|A diag(\omega_1, \dots, \omega_n) y_2\|_2$. Then*

$$\begin{aligned} \|\!|A|\!\|_2 &\ge \|A diag(\omega_1, \dots, \omega_n) y_2\|_2 = \|\!|A diag(\omega_1, \dots, \omega_n)|\!\|_2 \\ &\ge \|A diag(\omega_1, \dots, \omega_n) diag(\overline{\omega}_1, \dots, \overline{\omega}_n) y_1\|_2 = \|\!|A|\!\|_2, \end{aligned}$$

so we get our result.

For sub-probability measures $\{\mu_n\}$, μ, since for fixed $z \in \mathbb{C}^+$ the real and imaginary parts of $1/(x-z)$ are continuous and approach 0 as $|x| \to \infty$, we have $\mu_n \xrightarrow{v} \mu$ ($\xrightarrow{v}$ denoting vague convergence) implies $m_{\mu_n}(z) \to m_\mu(z)$. Conversely, if $m_{\mu_n}(z)$ converges for a countably infinite number of $z \in \mathbb{C}^+$ possessing a cluster point, all uniformly bounded away from the real axis, from Vitali's convergence theorem [[Titchmarsh, 1939, p. 168]], $m_{\mu_n}(z)$ converges for all z uniformly bounded away from the real axis to an analytic function m. Therefore any vaguely converging subsequence of $\{m_\mu\}$ has their Stieltjes transforms converging to m, and because of the existence of the inverse formula (1.3) we see that μ_n converges vaguely to a sub-probability measure μ having Stieltjes transform m. Thus we have

Lemma 2.20. *If for sub-probability measures μ_n, we have $m_{\mu_n}(z)$ converging for a countably infinite number of z uniformly bounded away from the real axis and possessing a cluster point, then there exists a sub-probability measure μ for which $\mu_n \xrightarrow{v} \mu$, or equivalently $D(\mu_n, \mu) \to 0$.*

Lemma 2.21 ([Shohat and Tamarkin, 1970, Lemma 2.2])**.** *Let f be analytic in $\mathbb{C}^+$ mapping $\mathbb{C}^+$ into $\mathbb{C}^+$, and there is a $\theta \in (0, \pi/2)$ for which $zf(z) \to c$, finite, as $z \to \infty$ restricted to $\{w \in \mathbb{C}^+ : \theta < \arg w < \pi - \theta\}$. Then f is the Stieltjes transform of a measure with total mass $-c$.*

Lemma 2.22 ([Horn and Johnson, 1990, Corolllary 8.1.20])**.** *Let C $N \times N$ have nonnegative entries. Then for each i $C_{ii} \le \rho(C)$.*

Lemma 2.23 ([Silverstein and Choi, 1995, Theorem 2.1])**.** *Let G be a probability distribution function and $x_0 \in \mathbb{R}$. Let m_G be its Stieltjes transform. Suppose $\Im m_G(x_0) \equiv \lim_{z \in \mathbb{C}^+} \Im m_G(z)$ exists. Then G is differentiable at x_o, and its derivative is $\frac{1}{\pi}\Im m_G(x_0)$.*

3. Proofs of the Theorem and Corollary

We begin by performing a series of truncations and centralizations on the entries of X_n and a truncation on the entries of D_n

Let

$$\widetilde{B}_n = \frac{1}{N}(D_n \circ \widetilde{X}_n)(D_n \circ \widetilde{X}_n)^*,$$

where

$$\widetilde{X}_{njk} = x_{jk}^{(n)} I(|x_{jk}^{(n)}| \le \eta_n \sqrt{n}).$$

Then from Lemma 2.5

$$\|F^{B_n} - F^{\widetilde{B}_n}\| \le \frac{1}{n} \sum_{jk} I(|x_{jk}^{(n)}| \ge \eta_n \sqrt{n}).$$

We have by (1.8)

$$\mathsf{E}\left(\frac{1}{n}\sum_{jk} I(|x_{jk}^{(n)}| \ge \eta_n \sqrt{n})\right) \le \frac{1}{\eta_n^2 n^2} \sum_{jk} \mathsf{E}|x_{jk}^{(n)}|^2 I(|x_{jk}^{(n)}| \ge \eta_n \sqrt{n}) = o(1),$$

and

$$\mathsf{Var}\left(\frac{1}{n}\sum_{jk} I(|x_{jk}^{(n)}| \ge \eta_n \sqrt{n})\right) \le \frac{1}{\eta_n^2 n^3} \sum_{jk} \mathsf{E}|x_{jk}^{(n)}|^2 I(|x_{jk}^{(n)}| \ge \eta_n \sqrt{n}) = o(1/n).$$

Therefore, from Lemma 2.6, for arbitrary positive ϵ we have for all n large

$$\mathsf{P}\left(\frac{1}{n}\sum_{jk} I(|x_{jk}^{(n)}| \geq \eta_n\sqrt{n}) > \epsilon\right) \leq \exp\left(\frac{-(\epsilon - o(1))^2}{2(o(1/n) + \frac{(1/n)(\epsilon - o(1))}{3})}\right)$$
$$\leq \exp\left(\frac{-n\epsilon^2}{8(1+\epsilon/3)}\right),$$

which is summable. Therefore, $\|F^{B_n} - F^{\widetilde{B}_n}\| \xrightarrow{a.s.} 0$ as $n \to \infty$.

For fixed $\epsilon > 0$, define $d = d_\epsilon \geq M_\epsilon$ and $\widetilde{d}_{jk}^n = \widetilde{d}_{jk}^{n\epsilon}$ as in the statement of Theorem 1.1, and let $\widetilde{D}_n = (\widetilde{d}_{jk}^n)$. From Lemma 2.4 we have

$$\|F^{\widetilde{B}_n} - F^{(1/N)(\widetilde{D}_n \circ \widetilde{X}_n)(\widetilde{D}_n \circ \widetilde{X}_n)^*}\| \leq \frac{1}{n} rank(\widetilde{X}_n \circ (D_n - \widetilde{D}_n))$$

The matrix $\widetilde{X}_n \circ (D_n - \widetilde{D}_n)$ has at most $\#E_{r\epsilon}^n$ nonzero rows and $\#E_{c\epsilon}^n$ columns. Therefore from Lemma 2.3 we have

$$\|F^{\widetilde{B}_n} - F^{(1/N)(\widetilde{D}_n \circ \widetilde{X}_n)(\widetilde{D}_n \circ \widetilde{X}_n)^*}\| \leq \epsilon.$$

By (1.1) and Lemma 2.1 we have

$$D^2\left(F_{sing}^{(1/\sqrt{N})\widetilde{D}_n \circ \widetilde{X}_n}, F_{sing}^{(1/\sqrt{N})\widetilde{D}_n \circ (\widetilde{X}_n - \mathsf{E}\widetilde{X}_n)}\right) \leq d^2 \frac{1}{nN} \sum_{jk} |\mathsf{E} x_{jk}^{(n)} I(|x_{jk}^{(n)} \leq \eta_n\sqrt{n})|^2$$
$$= d^2 \frac{1}{nN} \sum_{jk} |\mathsf{E} x_{jk}^{(n)} I(|x_{jk}^{(n)} > \eta_n\sqrt{n})|^2 \leq d^2 \frac{1}{nN} \sum_{jk} \mathsf{E}(|x_{jk}^{(n)}|^2 I(|x_{jk}^{(n)}| \geq \eta_n\sqrt{n}) \to 0$$

by (1.8). Since the set of subprobability measures is sequentially compact in vague topology with metric D and taking square roots of non-negative random variables is a continuous function, we have $D(F^{\widetilde{B}_n}, F^{(1/N)(\widetilde{D}_n \circ (\widetilde{X}_n - \mathsf{E}\widetilde{X}_n))(\widetilde{D}_n \circ (\widetilde{X}_n - \mathsf{E}\widetilde{X}_n))^*}) \to 0$ as $n \to \infty$.

Let

$$\widehat{X}_n = \left(\frac{x_{jk}^{(n)} I(|x_{jk}^{(n)}| \leq \eta_n\sqrt{n}) - \mathsf{E}(x_{jk}^{(n)} I(|x_{jk}^{(n)}| \leq \eta_n\sqrt{n}))}{\sigma_{jk}}\right),$$

where $\sigma_{jk}^2 = \sigma_{jk}^{2(n)} = \mathsf{E}|x_{jk}^{(n)} I(|x_{jk}^{(n)}| \leq \eta_n\sqrt{n}) - \mathsf{E}(x_{jk}^{(n)} I(|x_{jk}^{(n)}| \leq \eta_n\sqrt{n}))|^2$ (if $\sigma_{jk} = 0$, then define the corresponding entry of $\widehat{X}_n$ to be zero). Notice that $\sigma_{jk} \leq 1$ and, by (1.9), is $> \sqrt{f}$ for all n large. Then, again, by (1.1) and Lemma 2.1 we have

$$D^2(\left(F_{sing}^{(1/\sqrt{N})\widetilde{D}_n \circ \widehat{X}_n}, F_{sing}^{(1/\sqrt{N})\widetilde{D}_n \circ (\widetilde{X}_n - \mathsf{E}\widetilde{X}_n)}\right)$$
$$\leq d^2 \frac{1}{nN} \sum_{jk} (1 - \sigma_{jk}^{-1})^2 |x_{jk}^{(n)} I(|x_{jk}^{(n)}| \leq \eta_n\sqrt{n}) - \mathsf{E}(x_{jk}^{(n)} I(|x_{jk}^{(n)}| \leq \eta_n\sqrt{n}))|^2 \equiv d^2 a(n).$$

We have

$$\begin{aligned}\mathsf{E}a(n) &= \frac{1}{nN}\sum_{jk}(1-\sigma_{jk})^2 \le \frac{1}{nN}\sum_{jk}(1-\sigma_{jk}^2)\\ &= \frac{1}{nN}\sum_{jk}\mathsf{E}|x_{jk}^{(n)}|^2 I(|x_{jk}^{(n)}| > \eta_n\sqrt{n}) + |\mathsf{E}x_{jk}^{(n)} I(|x_{jk}^{(n)} > \eta_n\sqrt{n})|^2\\ &\qquad \le \frac{2}{nN}\sum_{jk}\mathsf{E}|x_{jk}^{(n)}|^2 I(|x_{jk}^{(n)}| \ge \eta_n\sqrt{n}) \to 0,\end{aligned}$$

as $n \to \infty$, by (2.1).

Let $a_{jk} = |x_{jk}^{(n)} I(|x_{jk}^{(n)}| \le \eta_n\sqrt{n}) - \mathsf{E}(x_{jk}^{(n)} I(|x_{jk}^{(n)}| \le \eta_n\sqrt{n}))|^2$. Using Lemma 2.7 and (1.9) for all n large we have

$$\begin{aligned}&\mathsf{E}|a(n)-\mathsf{E}a(n)|^4\\ &\le \frac{C_4}{(nN)^4}\left(\sum_{jk}(1-\sigma_{jk}^{-1}))^8\mathsf{E}|a_{jk}-\mathsf{E}a_{jk}|^4 + \left(\sum_{jk}(1-\sigma_{jk}^{-1}))^4\mathsf{E}|a_{jk}-\mathsf{E}a_{jk}|^2\right)^2\right)\\ &\quad \le \frac{C'}{n^8}\left(\sum_{jk}\mathsf{E}|x_{jk}^{(n)}|^8 I(|x_{jk}^{(n)}| \le \eta_n\sqrt{n}) + \left(\sum_{jk}\mathsf{E}|x_{jk}^{(n)}|^4 I(|x_{jk}^{(n)}| \le \eta_n\sqrt{n})\right)^2\right)\\ &\qquad \le \frac{C''}{n^8}(n^5\eta_n^6 + n^6\eta_n^4),\end{aligned}$$

which is summable.

Therefore, we conclude that

$$\limsup_n D(F^{B_n}, F^{(1/\sqrt{N})(\widetilde{D}_n\circ\widehat{X}_n)(\widetilde{D}_n\circ\widehat{X}_n)^*}) \le \epsilon \quad a.s. \tag{3.1}$$

From this point, for ease of notation, we will denote the augmented matrix also by $B_n = (1/\sqrt{N})(D_n \circ X_n)(D_n \circ X_n)^*$, where we will assume that $x_{jk}^{(n)}$, the entries of X_n satisfy

(1) For each n they are independent.
(2) $\mathsf{E}(x_{jk}^{(n)}) = 0$, $\mathsf{E}|x_{jk}^{(n)}|^2 = 1$.
(3) $|x_{jk}^{(n)}| \le \eta_n\sqrt{n}$,
where the present η_n are double the original η_n,

and that the elements of D_n are nonnegative and bounded by $d = d_\epsilon$. Keep in mind the difference between the two B_n's, along with the bound (3.1).

Let x_k denote the $k-$th column of X_n. Let D_k denote the $n \times n$ diagonal matrix consisting of the entries in the $k-$th column of D_n. Then we can write

$$B_n = \frac{1}{N}\sum_{k\le N} D_k x_k x_k^* D_k.$$

Let $\mathsf{E}_0(\cdot)$ denote expectation and $\mathsf{E}_k(\cdot)$ denote conditional expectation with respect to the σ-field generated by $x_1, \ldots, x_k$. Let for $k \le N$

$$B_{(k)} = \frac{1}{N}\sum_{j\ne k} D_j x_j x_j^* D_j.$$

Let $m_n(z)$, $z = x + iv$, $v > 0$ denote the Stieltjes transform of the eigenvalues of B_n. We have

$$m_n(z) = \frac{1}{n}\mathsf{tr}\,(B_n - zI)^{-1}.$$

The first step is to show

$$m_n(z) - \mathsf{E}m_n(z) \to 0 \quad \text{a.s. as } n \to \infty. \tag{3.2}$$

We have

$$m_n(z) - \mathsf{E}m_n(z) = \frac{1}{n}\sum_{k=1}^{N}[\mathsf{E}_k\mathsf{tr}\,(B_n - zI)^{-1} - \mathsf{E}_{k-1}\mathsf{tr}\,(B_n - zI)^{-1}]$$

$$= \frac{1}{n}\sum_{k=1}^{N}[\mathsf{E}_k(\mathsf{tr}\,(B_n-zI)^{-1}-\mathsf{tr}\,(B_{(k)}-zI)^{-1})-\mathsf{E}_{k-1}(\mathsf{tr}\,(B_n-zI)^{-1}-\mathsf{tr}\,(B_{(k)}-zI)^{-1})]$$

$$= \frac{1}{n}\sum_{k=1}^{N}\gamma_k.$$

By Lemma 2.10 we have each $|\gamma_k| \le 2/v$. Since the γ_k form a martingale difference sequence, we have by Lemma 2.8

$$\mathsf{E}|m_n(z) - \mathsf{E}m_n(z)|^4 \le \frac{C_p}{n^4}\mathsf{E}\left(\sum_{k=1}^{N}|\gamma_k|^2\right)^2 \le \frac{4C_pN^2}{v^4n^4},$$

which is summable. Therefore we have (3.2).

We turn to $\mathsf{E}m_n(z)$. Let

$$F = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{1+\frac{n}{N}e_k}D_k^2 \quad \text{where} \quad e_k = \frac{1}{n}\mathsf{E}\mathsf{tr}\,D_k(B_{(k)} - zI)^{-1}D_k.$$

Write

$$B_n - zI - (F - zI) = \frac{1}{N}\sum_{k=1}^{N}D_kx_kx_k^*D_k - F.$$

Taking inverses and using Lemma 2.11 we get

$$(F - zI)^{-1} - (B_n - zI)^{-1} \tag{3.3}$$

$$= \frac{1}{N}\sum_{k=1}^{N}\Bigg[\frac{1}{1+\frac{1}{N}x_k^*D_k(B_{(k)} - zI)^{-1}D_kx_k}(F - zI)^{-1}D_kx_kx_k^*D_k(B_{(k)} - zI)^{-1}$$

$$- (F - zI)^{-1}F(B_n - zI)^{-1}\Bigg]$$

Taking the trace, and dividing by n, we get

$$\frac{1}{n}\mathsf{tr}\,(F - zI)^{-1} - m_n(z) = \frac{1}{N}\sum_{k=1}^{N}\Bigg[\frac{\frac{1}{n}x_k^*D_k(B_{(k)} - zI)^{-1}(F - zI)^{-1}D_kx_k}{1+\frac{1}{N}x_k^*D_k(B_{(k)} - zI)^{-1}D_kx_k}$$

$$- \mathsf{tr}\,\frac{1}{n}(F - zI)^{-1}F(B_n - zI)^{-1}\Bigg]$$

$$= \frac{1}{N}\sum_{k=1}^{N}\left[\frac{\frac{1}{n}x_k^*D_k(B_{(k)} - zI)^{-1}(F - zI)^{-1}D_kx_k}{1+\frac{n}{N}\frac{1}{n}x_k^*D_k(B_{(k)} - zI)^{-1}D_kx_k} - \frac{\frac{1}{n}\mathsf{tr}\,(F - zI)^{-1}D_k^2(B_n - zI)^{-1}}{1+\frac{n}{N}e_k}\right]$$

$$= \frac{1}{N}\sum_{k=1}^{N} \alpha_k + \beta_k + \gamma_k,$$

where

$$\alpha_k = \frac{\frac{1}{n}x_k^* D_k(B_{(k)} - zI)^{-1}(F - zI)^{-1}D_k x_k}{1 + \frac{n}{N}\frac{1}{n}x_k^* D_k(B_{(k)} - zI)^{-1}D_k x_k} - \frac{\frac{1}{n}x_k^* D_k(B_{(k)} - zI)^{-1}(F - zI)^{-1}D_k x_k}{1 + \frac{n}{N}e_k},$$

$$\beta_k = \frac{\frac{1}{n}x_k^* D_k(B_{(k)} - zI)^{-1}(F - zI)^{-1}D_k x_k}{1 + \frac{n}{N}e_k} - \frac{\frac{1}{n}\operatorname{tr} D_k(B_{(k)} - zI)^{-1}(F - zI)^{-1}D_k}{1 + \frac{n}{N}e_k},$$

and

$$\gamma_k = \frac{\frac{1}{n}\operatorname{tr} D_k(B_{(k)} - zI)^{-1}(F - zI)^{-1}D_k}{1 + \frac{n}{N}e_k} - \frac{\frac{1}{n}\operatorname{tr}(F - zI)^{-1}D_k^2(B_n - zI)^{-1}}{1 + \frac{n}{N}e_k}.$$

Notice that the spectral norms of $(B_n - z)^{-1}$ and $(B_{(k)} - zI)^{-1}$ are bounded by $1/v$. Also notice that $\Im e_k > 0$ so that $\Im F < 0$. This implies $\|(F - zI)^{-1}\| < 1/v$.

We have $\mathsf{E}\beta_k = 0$. From Lemmas 2.9 and 2.10 we have

$$|\gamma_k| \le \frac{1}{n}\frac{d^2|z|}{v^3} \to 0.$$

as $n \to \infty$.

From Lemmas 2.9, 2.12, and the Cauchy-Schwarz inequality we have

$$|\mathsf{E}\alpha_k| \le \frac{n|z|^2}{Nv^2}((1/n^2)\mathsf{E}|x_k^* D_k(B_{(k)} - zI)^{-1}(F - zI)^{-1}D_k x_k|^2)^{1/2}$$
$$((1/n^2)(\mathsf{E}|x_k^* D_k(B_{(k)} - zI)^{-1}D_k x_k - \mathsf{E}\operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k|^2)^{1/2}.$$

We have

$$(3.4)\quad (1/n^2)\mathsf{E}|x_k^* D_k(B_{(k)} - zI)^{-1}(F - zI)^{-1}D_k x_k|^2 \le \frac{d_n^4}{v^4 n^2}\mathsf{E}\|x_k\|^4$$
$$= \frac{d^4}{v^4 n^2}\left(\sum_{j=1}^{n}\mathsf{E}|x_{jk}|^4 + n(n-1)\right) \le \frac{d^4}{v^4 n^2}(\eta_n^2 n^2 + n(n-1)) = O(1)$$

We have

$$((1/n^2)(\mathsf{E}|x_k^* D_k(B_{(k)} - zI)^{-1}D_k x_k - \mathsf{E}\operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k|^2$$
$$\le (2/n^2)(\mathsf{E}|x_k^* D_k(B_{(k)} - zI)^{-1}D_k x_k - \operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k|^2$$
$$+ (2/n^2)\mathsf{E}|\operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k - \mathsf{E}\operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k|^2.$$

Using Lemma 2.12 we have

$$(3.5)\quad (1/n^2)(\mathsf{E}|x_k^* D_k(B_{(k)} - zI)^{-1}D_k x_k - \operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k|^2$$
$$\le (C/n^2)(\eta_n^2 n)\mathsf{E}\operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k^2(B_{(k)} - \bar{z}I)^{-1}D_k$$
$$\le (C/n^2)(\eta_n^2 n)n\mathsf{E}\|D_k(B_{(k)} - zI)^{-1}D_k^2(B_{(k)} - \bar{z}I)^{-1}D_k\| \le C\eta_n^2 \to 0$$

as $n \to \infty$. As before, we can write $(1/n)(\operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k - \mathsf{E}\operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k)$ as a sum of martingale differences, where now each term is bounded in absolute value by $2d^2/v$. Using Lemma 2.8 we find then, that

$$(1/n^2)\mathsf{E}|\operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k - \mathsf{E}\operatorname{tr} D_k(B_{(k)} - zI)^{-1}D_k|^2 = O(1/n).$$

We conclude that

$$\mathsf{E}m_n(z) - (1/n)\mathsf{tr}\,(F - zI)^{-1} \to 0,$$

as $n \to \infty$.

In (3.3), multiply both sides on the left and right by D_j. Taking the trace and dividing by n we get

$$\frac{1}{n}\mathsf{tr}\,D_j(F-zI)^{-1}D_j - \frac{1}{n}\mathsf{tr}\,D_j(B_n - zI)^{-1}D_j$$
$$= \frac{1}{N}\sum_{k=1}^{N}\left[\frac{\frac{1}{n}x_k^* D_k(B_{(k)}-zI)^{-1}D_j^2(F-zI)^{-1}D_k x_k}{1+\frac{n}{N}\frac{1}{n}x_k^*D_k(B_{(k)}-zI)^{-1}D_k x_k} - \frac{\frac{1}{n}\mathsf{tr}\,D_j(F-zI)^{-1}D_k^2(B_n-zI)^{-1}D_j}{1+\frac{n}{N}e_k}\right].$$

In approaching this sum in the same way as before we can conclude that

$$\mathsf{E}(1/n)D_j(B_n - zI)^{-1}D_j - (1/n)\mathsf{tr}\,D_j(F-zI)^{-1}D_j \to 0$$

as $n \to \infty$. Using Lemma 2.10 we have

$$e_j(z) - (1/n)\mathsf{tr}\,D_j(F-zI)^{-1}D_j \to 0$$

as $n \to \infty$.

Thus we have, expressed in terms of the entries of D_n

$$m_n(z) - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\frac{1}{N}\sum_{k=1}^{N}\frac{d_{ik}^2}{1+\frac{n}{N}e_k(z)} - z} \to 0 \tag{3.6}$$

a.s. as $n \to \infty$, where the $e_j(z)$ satisfy

$$e_j(z) - \frac{1}{n}\sum_{i=1}^{n}\frac{d_{ij}^2}{\frac{1}{N}\sum_{k=1}^{N}\frac{d_{ik}^2}{1+\frac{n}{N}e_k(z)} - z} \to 0 \tag{3.7}$$

as $n \to \infty$.

We shall prove that for fixed n and N there exist $e_1^0(z), \ldots, e_N^0(z) \in \mathbb{C}^+$ that satisfies (1.6). Consider the matrix $B_{n\ell} = (1/(N\ell))(D^{(\ell)} \circ X^{(\ell)})(D^{(\ell)} \circ X^{(\ell)})^*$ where $X^{(\ell)}$ now is $n\ell \times N\ell$ containing i.i.d bounded standardized variables, and $D^{(\ell)}$ is $n\ell \times N\ell$ containing ℓ^2 copies of D_n. Then (3.7) holds a.s. as $\ell \to \infty$. Notice that for any positive integer $k \le N$ columns $k, N+k, \ldots, (\ell-1)N+k$ of $D^{(\ell)}$ are identical. Using Lemma 2.10 we see that

$$|e_{m_1 N+k}(z) - e_{m_2 N+k}(z)| \le \frac{2d^2}{n\ell v} \quad 0 \le m_1, m_2 \le \ell - 1.$$

Consider one realization for which (3.7) holds. Since the $e_k(z)$'s are bounded by d^2/v, we can find a subsequence of $\{\ell\}$ such that $e_1(z), \ldots, e_N(z)$ converge to $e_1^0(z), \ldots, e_N^0(z)$ on this subsequence. We have then for $1 \le i \le n\ell$

$$\frac{1}{N\ell}\sum_{k=1}^{N\ell}\frac{d_{ik}^{(\ell)2}}{1+\frac{n}{N}e_k(z)} = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{\ell}\sum_{m=1}^{\ell}\frac{d_{ik}^2}{1+\frac{n}{N}e_{mN+k}(z)} \to \frac{1}{N}\sum_{k=1}^{N}\frac{d_{ik}^2}{1+\frac{n}{N}e_k^0(z)},$$

on this subsequence. Since the $e_k(z)$ have positive imaginary part, the above limit has nonpositive imaginary part. Therefore the right side of (1.6) has positive imaginary part.

Fix $j \leq N$. Using the fact that the $d_{ij}^{(\ell)}$'s repeat as i ranges from 1 to $n\ell$, from (3.7) we get

$$e_j(z) - \frac{1}{n\ell}\sum_{i=1}^{n\ell} \frac{{d_{ij}^{(\ell)}}^2}{\frac{1}{N\ell}\sum_{k=1}^{N\ell}\frac{d_{ik}^{(\ell)2}}{1+\frac{n}{N}e_k(z)} - z} = e_j(z) - \frac{1}{n}\sum_{i=1}^{n} \frac{d_{ij}^2}{\frac{1}{N\ell}\sum_{k=1}^{N\ell}\frac{d_{ik}^{(\ell)2}}{1+\frac{n}{N}e_k(z)} - z} \to 0,$$

so we get (1.6) with each $e_j(z) \in \mathbb{C}_+$.

We will now show the uniqueness of the $e_k^0(z)$ in (1.6) having positive imaginary parts. Notice that from (1.6) the $e_k^0(z)$ are also bounded in absolute value by d^2/v. Let $e_{j,2}^0(z)$ denote the imaginary part of $e_j^0(z)$. Then

$$e_{j,2}^0(z) = \frac{1}{n}\sum_{i=1}^{n} d_{ij}^2 \frac{\frac{1}{N}\sum_{k=1}^{N} d_{ik}^2 \frac{\frac{n}{N}e_{k,2}^0(z)}{|1+\frac{n}{N}e_k^0(z)|^2} + v}{\left|\frac{1}{N}\sum_{\underline{k}=1}^{N}\frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z)} - z\right|^2}.$$

Let $C^0 = \{c_{jk}^0\}$, $N \times N$, and $b^0 = (b_1^0, \ldots, b_N^0)^T$, where

$$c_{jk}^0 = \frac{1}{N^2}\sum_{i=1}^{n} d_{ij}^2 d_{ik}^2 \frac{\frac{1}{|1+\frac{n}{N}e_k^0(z)|^2}}{\left|\frac{1}{N}\sum_{\underline{k}=1}^{N}\frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z)} - z\right|^2}$$

and

$$b_j^0 = \frac{1}{n}\sum_{i=1}^{n} \frac{d_{ij}^2}{\left|\frac{1}{N}\sum_{\underline{k}=1}^{N}\frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z)} - z\right|^2}.$$

Let $e_2^0 = (e_{1,2}^0(z), \ldots, e_{N,2}^0(z))^T$. Then

$$e_2^0 = C^0 e_2^0 + v b^0. \tag{3.8}$$

Notice because of the nonzero assumption on any column of D_n, equation (3.8) has all components of $e_2^0(z)$ and b^0 positive

Suppose $\rho(C^0)$, the spectral radius of C^0, is greater than or equal to 1. Then by Lemma 2.13 $\rho(C^0)$ is an eigenvalue of C^0 with left eigenvector y^T containing nonnegative entries, and necessarily, at least one entry is positive. Multiplying on the left of both sides of (3.8) by y^T we get

$$y^T e_2^0 = \rho(C^0) y^T e_2^0 + v y^T b^0. \tag{3.9}$$

Since both $y^T e_2^0$ and $y^T b^0$ are positive, we arrive at a contradiction. Therefore

$$\rho(C^0) < 1. \tag{3.10}$$

Suppose $\underline{e}_1^0(z), \ldots, \underline{e}_1^0(z)$ also satisfy (1.6). Let $\underline{e}_2^0$ and $\underline{C}^0 = \{\underline{c}_{jk}^0\}$ denote the quantities analgous to e_2^0 and C^0. Then for nonzero $e_j^0(z)$ and $\underline{e}_j^0$(z) we have

$$e_j^0 - \underline{e}_j^0 = \sum_{k=1}^{N} a_{jk}(e_k^0(z) - \underline{e}_k^0(z)),$$

where

$$a_{jk} = \frac{1}{N^2}\frac{1}{(1+\frac{n}{N}e_k^0(z))(1+\frac{n}{N}\underline{e}_k^0(z))} \times \sum_{i=1}^n \frac{d_{ij}^2 d_{ik}^2}{\left(\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z)} - z\right)\left(\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{i\underline{k}}^2}{1+\frac{n}{N}\underline{e}_{\underline{k}}^0(z)} - z\right)}. \tag{3.11}$$

Therefore, with $e^0 = (e_1^0(z), \ldots, e_N^0(z))^T$ and $\underline{e}^0 = (\underline{e}_1^0(z), \ldots, \underline{e}_N^0(z))^T$ we have

$$e^0 - \underline{e}^0 = A(e^0 - \underline{e}^0), \tag{3.12}$$

where $A = (a_{jk})$. If $e^0 \neq \underline{e}^0$ then A has an eigenvalue equal to 1. However applying Cauchy-Schwarz we see that

$$|a_{jk}| \leq {c_{jk}^0}^{\frac{1}{2}} {\underline{c}_{jk}^0}^{\frac{1}{2}},$$

and applying Lemmas 2.14 and 2.15

$$\rho(A) \leq \rho({c_{jk}^0}^{\frac{1}{2}} {\underline{c}_{jk}^0}^{\frac{1}{2}}) \leq (\rho(C^0))^{\frac{1}{2}}(\rho(\underline{C}^0))^{\frac{1}{2}} < 1,$$

by (3.10), a contradiction. Therefore we have $e^0 = \underline{e}^0$.

Using the last part of the above argument, we will show that $e^0(z)$ is a continuous function of $z \in \mathbb{C}^+$. Let $\{z_n\}$ and $\{z_n'\}$ be two sequences in $\mathbb{C}^+$ each converging to $z \in \mathbb{C}^=$. Let $e^0(z) = (e_1^0(z), \ldots, e_N^0(z))^T$. Then $e^0(z_n)$ and $e^0(z_n')$ satisfy

$$e^0(z_n) - e^0(z_n') = A(z_n, z_n')(e^0(z_n) - e^0(z_n')) + (z_n - z_n')b(z_n, z_n'), \tag{3.13}$$

where

$$A(z_n, z_n')_{jk} = \frac{1}{N^2}\frac{1}{(1+\frac{n}{N}e_k^0(z_n))(1+\frac{n}{N}e_k^0(z_n'))} \sum_{i=1}^n \frac{d_{ij}^2 d_{ik}^2}{\left(\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z_n)} - z_n\right)\left(\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z_n')} - z_n'\right)}.$$

and

$$b(z_n, z_n')_j = \frac{1}{n}\sum_{i=1}^n \frac{d_{ij}^2}{\left(\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z_n)} - z_n\right)\left(\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z_n')} - z_n'\right)}.$$

Assume $e^0(z_n) \to e^0$ and $e^0(z_n') \to \underline{e}^0$ as $n \to \infty$. Since all quantities are bounded we get as $n \to \infty$ (3.12), and therefore $e^0 = \underline{e}^0$, which proves continuity of $e^0(z)$, $z \in \mathbb{C}^+$.

We claim that each $e_k(z)$ is the Stieltjes transform of a measure with total mass $(1/n)\mathsf{tr}\, D_k^2$. Indeed, for all z with imaginary part $\geq v > 0$ any difference quotient

$$\frac{\frac{1}{n}\mathsf{tr}\, D_k(B_{(k)} - z_1 I)^{-1}D_k - \frac{1}{n}\mathsf{tr}\, D_k(B_{(k)} - z_2 I)^{-1}D_k}{z_1 - z_2} = \frac{1}{n}\mathsf{tr}\, D_k(B_{(k)} - z_1 I)^{-1}(B_{(k)} - z_2 I)^{-1}D_k$$

is uniformly bounded. Therefore, by the dominated convergence theorem e_k is differentiable for all $z \in \mathbb{C}^+$, and hence is analytic on $\mathbb{C}^+$. Moreover it is clear that

e_k maps $\mathbb{C}^+$ into $\mathbb{C}^+$ and, again by the dominated convergence theorem, the limit of $ze_k(z)$ as $z \to \infty$ is $-(1/n)\mathsf{tr}\, D_k^2$. Therefore by Lemma 2.21 the claim is proven.

Now, since, from the above existence argument, for each n, e_k^0 is the limit of a sequence of e_k's we have, using the Helly selection theorem, e_k^0 is itself the Stieltjes transform of a measure with total mass $(1/n)\mathsf{tr}\, D_k^2$. Therefore the function $G_n(z)$ is analytic in $\mathbb{C}^+$, maps $\mathbb{C}^+$ to $\mathbb{C}^+$, and as $z \to \infty$, $zG_n(z) \to -1$. Therefore, by Lemma 2.21 $G_n(z)$ is the Stieltjes transform of a probability measure, F_n^0, on $\mathbb{R}$.

We will proceed to show that the $e_k(z)$ in (3.6) can be replaced with the $e_k^0(z)$, that is, we will show that

$$m_n(z) - G_n(z) = m_n(z) - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\frac{1}{N}\sum_{k=1}^{N} \frac{d_{ik}^2}{1+\frac{n}{N}e_k^0(z)} - z} \to 0 \tag{3.14}$$

almost surely. We have

$$\frac{1}{n}\sum_{i=1}^{n} \frac{1}{\frac{1}{N}\sum_{k=1}^{N} \frac{d_{ik}^2}{1+\frac{n}{N}e_k(z)} - z} - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\frac{1}{N}\sum_{k=1}^{N} \frac{d_{ik}^2}{1+\frac{n}{N}e_k^0(z)} - z} \tag{3.15}$$

$$= \frac{1}{N^2}\sum_{k=1}^{N} \frac{(e_k(z) - e_k^0(z))}{(1+\frac{n}{N}e_k(z))(1+\frac{n}{N}e_k^0(z))}$$
$$\times \sum_{i=1}^{n} \frac{d_{ik}^2}{\left(\frac{1}{N}\sum_{\underline{k}=1}^{N} \frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}(z)} - z\right)\left(\frac{1}{N}\sum_{\underline{k}=1}^{N} \frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z)} - z\right)}.$$

Returning now to C^0, b^0, and e_2^0, we see that the entries of b^0 are uniformly bounded. Let $\mathbf{1}_m$ denote the m dimensional vector containing all one's, and let "$\leq$" between two vectors denote entry wise $\leq$. Using Lemma 2.9 we see that

$$C^0\mathbf{1}_N \leq \left(\frac{n}{N}\frac{|z|^2}{v^2}\max_i \frac{1}{N}\sum_{\underline{k}=1}^{N} d_{i\underline{k}}^2\right) b^0. \tag{3.16}$$

Since the entries of e_2^0 are uniformly bounded we see from (3.8) that

$$e_2^0 \leq k_1 b^0. \tag{3.17}$$

From (3.9), where y is the nonnegative eigenvector of C^0 associated with $\rho(C^0)$ we get

$$1 - \rho(C^0) = \frac{vy^T b^0}{y^T e_2^0} \geq k, \tag{3.18}$$

where necessarily $k \in (0,1)$. Let $\omega \in \mathbb{R}^n$ have for its j-th entry the imaginary part of the lefthand side of (3.7). From condition (1.9) we see that η_n approaches zero more slowly than $1/n$. Thus, from the derivation of (3.7) it is clear that

$$|\omega| \leq k_1\eta_n\mathbf{1}_N. \tag{3.19}$$

Let $e_{j,2}(z)$ denote the imaginary part of $e_j(z)$ and $e_2 = (e_{1,2}, \ldots, e_{N,2})^T$. Similar to C^0 and b^0 we have

$$e_2 = Ce_2 + vb + \omega, \tag{3.20}$$

where

$$c_{jk} = \frac{1}{N^2}\sum_{i=1}^{n} d_{ij}^2 d_{ik}^2 \frac{\frac{1}{|1+\frac{n}{N}e_k(z)|^2}}{\left|\frac{1}{N}\sum_{\underline{k}=1}^{N}\frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}(z)} - z\right|^2}$$

and

$$b_j = \frac{1}{n}\sum_{i=1}^{n}\frac{d_{ij}^2}{\left|\frac{1}{N}\sum_{\underline{k}=1}^{N}\frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}(z)} - z\right|^2}.$$

We will show

$$\limsup_{n\to\infty}\rho(C) \le 1. \tag{3.21}$$

Rather than work with C we consider $C_s \equiv E^{-1}CE$ where $E = \text{diag}(|1 + \frac{n}{N}e_1(z)|, \ldots, |1+\frac{n}{N}e_N(z)|)$. Notice C_s is a symmetric matrix and its eigenvalues are the same as C. From Lemma 1.7 and the fact that $|1+\frac{n}{N}e_j(z)| \le 1+\frac{1}{Nv}\sum_{i=1}^{n} d_{ij}^2$, we see that the diagonal entries of E and E^{-1} are uniformly bounded. Let $f = E^{-1}e_2$, $\underline{b} = E^{-1}vb$, and $\underline{\omega} = E^{-1}\omega$. Then from (3.20) we have

$$f = C_s f + \underline{b} + \underline{\omega}. \tag{3.22}$$

We have the entries of $\underline{b}$ uniformly bounded,

$$|\underline{\omega}| \le k_2\eta_n \mathbf{1}_N$$

and, similar to (3.16), we have

$$C_s\mathbf{1}_N \le k_3\underline{b}. \tag{3.23}$$

Consider those entries of f_i for which $\underline{b}_i + \underline{\omega}_i > 0$ and those entries for which $\underline{b}_i + \underline{\omega}_i$ are negative. Rearrange the coordinates so that the first ℓ entries satisfy the former, the remaining the latter and put C_s into corresponding block form:

$$C_s = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Splitting f, $\underline{b}$ and $\underline{\omega}$ into appropriate parts we have

$$f_1 = C_{11}f_1 + C_{12} + \underline{b}_1 + \underline{\omega}_1 \tag{3.24}$$

$$f_2 = C_{21}f_1 + C_{22}f_2 + \underline{b}_2 + \underline{\omega}_2, \tag{3.25}$$

where $f = (f_1^T, f_2^T)^T$, etc. Applying the left nonnegative eigenvector of C_{11} corresponding to $\rho(C_{11})$ to both sides of (3.24) we see that $\rho(C_{11}) < 1$. There exists k_4 for which

$$C_{11}\mathbf{1}_\ell \le k_4\mathbf{1}_\ell,\ C_{12}\mathbf{1}_{N-\ell} \le k_4\mathbf{1}_\ell,\ \underline{b}_2 \le k_4\eta_n\mathbf{1}_{N-\ell},\ C_{21}\mathbf{1}_\ell \le k_4\eta_n\mathbf{1}_{N-\ell},\ \text{and } C_{22}\mathbf{1}_{N-\ell} \le k_4\eta_n\mathbf{1}_{N-\ell}. \tag{3.26}$$

From Lemma 2.17 we see that $\rho(C_{22}) \le k_4\eta_n$, which is less than $1/2$ for all n large. Notice then for these n (which we assume we consider from this point on) that if $\ell = 0$ or N then $\rho(C) = \rho(C_s) < 1$ and we would be done. So we assume $1 < \ell < N$.

Let $x = (x_1^T x_2^T)^T$ be a nonnegative eigenvector of of C_s corresponding to $\rho(C_s)$. Then we have

$$\rho(C_s)x_1 = C_{11}x_1 + C_{12}x_2$$
$$\rho(C_s)x_2 = C_{21}x_1 + C_{22}x_2.$$

When $\rho(C_s) > 1$ we have $\rho(C_s)I - C_{22}$ invertible (smallest eigenvalue $\geq 1/2$ and $\|(\rho(C_s)I - C_{22})^{-1}\| \leq 2$. From the second identity we have $x_2 = (\rho(C_s) - C_{22})^{-1}C_{21}x_1$ and plugging this into the first we find

$$\rho(C_s)x_1 = (C_{11} + C_{12}(\rho(C_s)I - C_{22})^{-1}C_{21})x_1. \tag{3.27}$$

We have when $\rho(C_s) \geq 1$

$$(\rho(C_s)I - C_{22})^{-1}\mathbf{1}_{N-\ell} = \frac{1}{\rho(C_s)}\sum_{m=0}^{\infty}\frac{C_{22}^m}{\rho(C_s)^m}\mathbf{1}_{N-\ell}$$

$$\leq \frac{1}{\rho(C_s)}\sum_{m=1}^{\infty}\frac{1}{(2\rho(C_s))^m}\mathbf{1}_{N-\ell} = \frac{1}{\rho(C_s)}\frac{1}{1-\frac{1}{2\rho(C_s)}}\mathbf{1}_{N-\ell} = \frac{1}{\rho(C_s)-\frac{1}{2}}\mathbf{1}_{N-\ell}.$$

From (3.26) we have

$$C_{12}(\rho(C_s)I - C_{22})^{-1}C_{21}\mathbf{1}_\ell \leq \frac{\eta_n k_4^2}{\rho(C_s)-\frac{1}{2}}\mathbf{1}_\ell.$$

Therefore, for all n large, when $\rho(C_s) \geq 1$

$$\rho(C_{12}(\rho(C_s)I - C_{22})^{-1}C_{21}) = \left|\left|\left|C_{12}(\rho(C_s)I - C_{22})^{-1}C_{21}\right|\right|\right|_2 \leq k_5\eta_n \tag{3.28}$$

If $x_1 = 0$, then $\rho(C_s) = \rho(C_{22}) \leq 1/2$. Therefore, when $\rho(C_s) \geq 1$ we have $x_1 \neq 0$ and so applying the spectral norm on (3.27) we have

$$\rho(C_s) \leq \rho(C_{11}) + k_5\eta_n,$$

and so (3.21) holds.

Let $e = (e_1(z), \dots, e_N(z))^T$. Let now $\omega \in \mathbb{C}^n$ have for its entries the lefthand side of (3.7). We may assume

$$|\omega| \leq k_4\eta_n\mathbf{1}_N. \tag{3.29}$$

From (1.6) and (3.7), we have

$$e - e^0 = A(e - e^0) + \omega,$$

where $A = (a_{jk})$ with

$$a_{jk} = \frac{1}{N^2}\frac{1}{(1+\frac{n}{N}e_k(z))(1+\frac{n}{N}e_k^0(z))}$$
$$\sum_{i=1}^{n}\frac{d_{ij}^2 d_{ik}^2}{\left(\frac{1}{N}\sum_{\underline{k}=1}^{N}\frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}(z)} - z\right)\left(\frac{1}{N}\sum_{\underline{k}=1}^{N}\frac{d_{i\underline{k}}^2}{1+\frac{n}{N}e_{\underline{k}}^0(z)} - z\right)}.$$

Using the same arguments seen earlier, we have from (3.18), (3.21) for all n large the existence of a $k_6 \in (0,1)$ such that

$$\rho(A) \leq k_6.$$

Therefore I-A is invertible and we get

$$e - e^0 = (I - A)^{-1}\omega. \tag{3.30}$$

Let $D_2 = D_n \circ D_n$, $F^0 = \text{diag}(1 + \frac{n}{N}e_1^0(z), \ldots, 1 + \frac{n}{N}e_N^0(z))$, $F = \text{diag}(1 + \frac{n}{N}e_1(z), \ldots, 1 + \frac{n}{N}e_N(z))$,

$$G^0 = \text{diag}\left(\frac{1}{\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{1\underline{k}}^2}{1+\frac{n}{M}e_{\underline{k}}^0(z)} - z}, \ldots, \frac{1}{\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{n\underline{k}}^2}{1+\frac{n}{M}e_{\underline{k}}^0(z)} - z}\right),$$

and

$$G = \text{diag}\left(\frac{1}{\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{1\underline{k}}^2}{1+\frac{n}{M}e_{\underline{k}}(z)} - z}, \ldots, \frac{1}{\frac{1}{N}\sum_{\underline{k}=1}^N \frac{d_{n\underline{k}}^2}{1+\frac{n}{M}e_{\underline{k}}(z)} - z}\right).$$

Then we can write $A = \frac{1}{N^2}D_2^T GG^0 D_2 {F^0}^{-1} F^{-1}$. Writing $F^{-1} = E^{-1}\text{diag}(\omega_1, \ldots, \omega_N)$ where the ω_i's are on the unit circle in the complex plane, we see that since $((1/N)GD_2E^{-1})^*(1/N)GD_2E^{-1} = C_s$, we have from Lemma 2.19 $|||(1/N)GD_2F^{-1}|||_2 = \rho^{1/2}(C)$. Similarly we have $\left|\left|\left|(1/N)G^0D_2{F^0}^{-1}\right|\right|\right|_2 = \rho^{1/2}(C^0)$. Therefore, for all n large we have $k_7 \in (0,1)$ such that

$$|||F^{-1}AF|||_2 \leq |||(1/N)GD_2F^{-1}|||_2 \left|\left|\left|(1/N)G^0D_2{F^0}^{-1}\right|\right|\right|_2 < k_7,$$

and so $(I - F^{-1}AF)^{-1}$ exists and is bounded in spectral norm for all n large. Therefore, using the fact that F is bounded in spectral norm we have

$$e - e^0 = F(I - F^{-1}AF)^{-1}F^{-1}\omega = H\omega,$$

where H is bounded in spectral norm for all n large.

We have then (3.15) $= \frac{1}{N}\mathbf{1}_N^T GG^0(1/N)D_2F^{-1}{F^0}^{-1}H\omega$. It is straightforward to verify that the entries of the vector $\mathbf{1}_N^T GG^0(1/N)D_2F^{-1}{F^0}^{-1}$ are bounded for all large n. Using the fact that $\|\mathbf{1}_N\|_2 = \sqrt{N}$ along with (3.29) we conclude that

$$\left|\frac{1}{N}\mathbf{1}_N^T GG^0(1/N)D_2F^{-1}{F^0}^{-1}H\omega\right| \leq \frac{1}{N}\|\mathbf{1}_N^T GG^0(1/N)D_2F^{-1}{F^0}^{-1}\|_2 |||H|||_2 \|\omega\|_2$$
$$\leq k_8\eta_n \to 0$$

as $n \to \infty$. Therefore we get (3.14).

We proceed to complete the proof of the theorem. With probability one, say on the set A, (3.14) holds for a countably infinite collection $\{z_m\}$ of $z \in \mathbb{C}^+$ uniformly bounded away from the real axis having a cluster point. For any $\omega \in A$, and any vaguely converging subsequence of $F^{B_{n_j}}$, say to F_μ with μ a sub-probability measure, we have $m_{n_j}(z) \to m_\mu(z) \equiv \int \frac{1}{x-z}dF_\mu$, $z \in \{z_m\}$. Necessarily $G_{n_j}(z) \to m_\mu(z)$, $z \in \{z_m\}$. By Lemma 2.20 we have $F_{n_j}^0$ converging vaguely to the distribution function of a measure, which, because of uniqueness of measures and their Stieltjes transforms, must necessarily be μ. Thus, $D(F^{B_{n_j}}, F_{n_j}^0) \to 0$ on this subsequence. Since by the Helly selection theorem for an arbitrary subsequence, there exists a further subsequence for which vague convergence holds, we must have

$$D(F^{B_n}, F_n^0) \to 0, \quad \omega \in A.$$

Combining this result with (3.1) we have for any $\epsilon > 0$ with probability one

$$\limsup_n D(F^{B_n}, F_{n,\epsilon}^0) \leq \epsilon,$$

where B_n is now the original matrix defined in (1.1) and $F_{n,\epsilon}$ is the distribution function of the probability measure having Stieltjes transform $G_{n,\epsilon}$, which is G_n defined in terms of the truncated d_{jk}^n's, namely $d_{jk}^n I(d_{jk}^n \leq d_\epsilon)$ with $d_\epsilon \geq M_\epsilon$.

For the proof of the corollary, we let A be a set of probability one for which for each $\omega \in A$

$$\limsup_n D(F^{B_n}, F^0_{n,1/m}) \leq 1/m, \quad m = 1, 2, \ldots.$$

For fixed $\omega \in A$, choose integers $N_2 > N_1 > 0$ arbitrarily. Choose integer $N_3 > N_2$ for which $D(F^{B_n}, F^0_{n,1/3}) \leq 2/3$ for all $n \geq N_3$, and recursively choose $N_m > N_{m-1}$ for which $D(F^{B_n}, F^0_{n,1/m}) \leq 2/m$ for all $n \geq N_m$. We thus have the existence of $\{\epsilon_n\}$, such that $D(F^{B_n}, F^0_{n,\epsilon_n}) \to 0$, an event which occurs with probability one.

We conclude with a way to compute e^0 associated with $D_n = (d_{ij})$. We will show that there exists a neighborhood of e^0 for which the scheme

$$e_j^{\ell+1} = \frac{1}{n} \sum_{i=1}^n \frac{d_{ij}^2}{\frac{1}{N} \sum_{k=1}^N \frac{d_{ik}^2}{1 + \frac{n}{N} e_k^\ell(z)} - z}. \tag{3.31}$$

converges to e^0. Consider the matrix A in (3.12) with $\underline{e}^0$ replaced by e and denote it by $A(e)$. Then we have $\rho(A(e^0)) < 1$. By Lemma 2.18 we can find a vector norm $\|\cdot\|$ where its induced matrix norm $|||\cdot|||$ satisfies $\left|\left|\left|A(e^0)\right|\right|\right| \leq \alpha < 1$ for some α. Then, for a given $\beta \in (\alpha, 1)$, by continuity we can find a $\|\cdot\|$ open ball $\mathbb{B}$ of e^0 such that $|||A(e)||| \leq \beta$ for all $e \in \mathbb{B}$. Therefore, writing $e^\ell = (e_1^\ell(z), \ldots, e_N^\ell(z))^T$, $e^{\ell+1} = (e_1^{\ell+1}(z), \ldots, e_N^{\ell+1}(z))^T$, if $e^\ell \in \mathbb{B}$ we have

$$e^0 - e^{\ell+1} = A(e^\ell)(e^0 - e^\ell),$$

and

$$\|e^0 - e^{\ell+1}\| \leq \left|\left|\left|A(e^\ell)\right|\right|\right| \|e^0 - e^\ell\| \leq \beta |\|e^0 - e^\ell\|.$$

Therefore $e^{\ell+1} \in \mathbb{B}$ and we get convergence to e^0.

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References

[Bai and Silverstein, 2010] Bai, Z. and Silverstein, J. W. (2010). *Spectral analysis of large dimensional random matrices.* Springer.

[Bai and Silverstein, 1998] Bai, Z. D. and Silverstein, J. W. (1998). No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. *The Annals of Probability*, 26(1):316–345.

[Girko 2001] Girko, V. L. (2001) *Theory of Stochastic Canonical Equations Vol. 1* Kluwer Academic Publishers, Dordrecht/Boston/London.

[Horn and Johnson, 1990] Horn, R. A. and Johnson, C. R. (1990). *Matrix analysis.* Cambridge University Presss, Cambridge.

[Horn and Johnson, 1991] Horn, R. A. and Johnson, C. R. (1991). *Topics in matrix analysis.* Cambridge University Press, Cambridge.

[Shohat and Tamarkin, 1970] Shohat, J. A. and Tamarkin, J. D. (1970) *The Problem of Moments* American Mathematical Society, Providence.

[Silverstein and Choi, 1995] Silverstein, J. W. and Choi, S. I. (1995). Analysis of the limiting spectral distribution of large dimensional random matrices. *Journal of Multivariate Analysis* 54(2):295 – 309.

[Titchmarsh, 1939] Titchmarsh, E. C. (1939). *The Theory of Functions* Oxford University Press

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