Analysis of the Limiting Spectral Distribution of Large Dimensional Information-Plus-Noise Type Matrices

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Abstract

A derivation of results on the analytic behavior of the limiting spectral distribution of sample covariance matrices of the "information-plus-noise" type, as studied in Dozier and Silverstein [3], is presented. It is shown that, away from zero, the limiting distribution possesses a continuous density. The density is analytic where it is positive and, for the most relevant cases of a in the boundary of its support, exhibits behavior closely resembling that of $\sqrt{|x-a|}$ for x near a. A procedure to determine its support is also analyzed.

1. Introduction

For $n = 1, 2, \ldots$ and $N = N(n)$ let $C_n = \frac{1}{N}(R_n + \sigma X_n)(R_n + \sigma X_n)^*$, where $X_n = (X_{ij}^n)$ is $n \times N$, $X_{ij}^n \in \mathbb{C}$, identically distributed for all n, i, j, independent across i, j for each n, $E|X_{11}^1 - EX_{11}^1|^2 = 1$, $\frac{n}{N} \to c > 0$ as $n \to \infty$, $\sigma > 0$ is constant, and R_n is an $n \times N$ random matrix independent of X_n. For any square matrix A with only real eigenvalues, let F^A denote the empirical distribution function (e.d.f.) of the eigenvalues of A. Assume $F^{\frac{1}{N}R_nR_n^*} \xrightarrow{\mathcal{D}} H$, a.s., where H is a nonrandom probability distribution function (p.d.f.). Then it is shown in Dozier and Silverstein [3] that, almost surely, $F^{C_n} \xrightarrow{\mathcal{D}} F$, where F is a nonrandom p.d.f. which depends on H, c, and σ. The aim of the present paper is to derive analytic properties of F.

The matrix C_n can be thought of as the sample correlation matrix of N samples of the form $R_{\cdot i} + \sigma X_{\cdot i}$, where the $n \times 1$ vectors $R_{\cdot i}$ are stationary ergodic with correlation matrix $S_n \equiv ER_{\cdot 1}R_{\cdot 1}^*$ and the $X_{\cdot i}$'s represent components of additive noise (variance σ^2 unknown) that corrupt the $R_{\cdot i}$'s. If the noise is centered ($EX_{11} = 0$), and N is sufficiently large, then C_n provides a reasonable estimate of $S_n + \sigma^2 I$ (I denoting the identity matrix), which would reveal S_n, if S_n were known to be singular. However, if n is large, then the number of samples needed to provide an adequate approximation of $S_n + \sigma^2 I$ is unattainable. As in Dozier and Silverstein [3], our assumption $\frac{n}{N} \to c > 0$ models the situation of sample size and vector dimension being on the same order of magnitude.

An area in which our results have significance is that of the detection problem in array signal processing, that is, the problem of observing data collected at n sensors which receive signals transmitted from an unknown number of sources in a noise-filled environment, and using

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this data to determine the number of sources. The importance of such results to array signal processing is discussed in Silverstein and Combettes [10], however, in a less general setting. In that paper certain internal independence assumptions are imposed upon the signal matrix R_n, specifically, independence across samples is assumed. In this paper, as in Dozier and Silverstein [3], we require only that, almost surely, the e.d.f. of the eigenvalues of $\frac{1}{N}R_nR_n^*$ converges in distribution to some nonrandom p.d.f. H, thus allowing the detection problem to be studied under more general settings. Further details on the detection problem are presented in the last section of this paper along with a discussion of the applicability of results in the theory of large dimensional random matrices.

The work done in Dozier and Silverstein [3] relies heavily on Stieltjes transforms of measures. For any p.d.f. G, the Stieltjes transform of G is defined by

$$m_G(z) = \int \frac{dG(\lambda)}{\lambda - z} \qquad \text{for} \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \mathcal{I}m\, z > 0\},$$

and we may retrieve G by the inversion formula

$$G\{[a,b]\} = \frac{1}{\pi} \lim_{\eta \to 0^+} \int_a^b \mathcal{I}m\, m_G(\xi + i\eta) d\xi,$$

where a, b are continuity points of G. It is shown in Dozier and Silverstein [3] that $m = m_F(z)$, the Stieltjes transform of the limiting spectral distribution of C_n, satisfies the equation

$$m = \int \frac{dH(t)}{\frac{t}{1+\sigma^2 cm} - (1+\sigma^2 cm)z + \sigma^2(1-c)} \tag{1.1}$$

for any $z \in \mathbb{C}^+$, and it is the unique solution $m \in \mathbb{C}^+$ for which $\mathcal{I}m\, mz \geq 0$. This equation and the fact that m is a Stieltjes transform reveal much of the behavior of F. A useful property of Stieltjes transforms is that if G is any p.d.f. whose support is nonnegative, then for any $z \in \mathbb{C}^+$

$$\mathcal{I}m\, m_G(z)z \geq 0. \tag{$S.1$}$$

Therefore, using (S.1), we have for all $t \geq 0$

$$\mathcal{I}m \left(\frac{t}{1+\sigma^2 cm} - (1+\sigma^2 cm)z + \sigma^2(1-c) \right) \leq -\mathcal{I}m\, z < 0.$$

Hence, for any $z \in \mathbb{C}^+$, the integral in (1.1) is well-defined.

We note that it is shown in Silverstein [8] that, almost surely, $F^{\frac{1}{N}\sigma^2 X_n X_n^*}$ converges in distribution to a nonrandom p.d.f. F_* whose Stieltjes transform $m_* = m_{F_*}(z)$, for $z \in \mathbb{C}^+$, satisfies the equation

$$m_* = \frac{1}{\sigma^2(1-c-czm_*) - z} = \frac{1}{-(1+\sigma^2 cm_*)z + \sigma^2(1-c)},$$

which is equation (1.1) with $H = 1_{[0,\infty)}$ (1_B denoting the indicator function over the set B). Therefore, by uniqueness of solution (Theorem 4.1 of Dozier and Silverstein [3]), we have $m_* = m_F$ (for $H = 1_{[0,\infty)}$), and hence $F = F_*$. This function has an explicit expression (Marčenko and Pastur [5]), satisfying all properties to be investigated in this paper. Therefore for the rest of this paper we may assume $H \neq 1_{[0,\infty)}$.

Let $\mathbf{C} = \frac{1}{N}(R_n + \sigma X_n)^*(R_n + \sigma X_n)$. The spectra of C_n and $\mathbf{C}_n$ differ by $|n - N|$ zero eigenvalues and is expressed in

$$F^{\mathbf{C}_n} = \left(1 - \frac{n}{N}\right) 1_{[0,\infty)} + \frac{n}{N} F^{C_n}.$$

Using this and writing $\frac{1}{n}(R_n^* + \sigma X_n^*)(R_n^* + \sigma X_n^*)^* = \frac{N}{n}\mathbf{C}_n$, it is straightforward to show that if m_F satisfies (1.1) when $c \leq 1$, then m_F will satisfy (1.1) when $c > 1$. Therefore, without loss of generality, we assume, as in Dozier and Silverstein [3], that $c \leq 1$.

We see from equation (1.1) that if $c \downarrow 0$, we get for any $z \in \mathbb{C}^+$

$$m_F(z) \longrightarrow \int \frac{dH(t)}{(t + \sigma^2) - z},$$

which is the Stieltjes transform of the p.d.f. of a random variable $Y + \sigma^2$, where Y has distribution H. In terms of the aforementioned application to array signal processing, the condition $c \downarrow 0$ corresponds to the situation when the number of samples, N, is significantly larger than the number of sensors, n, and, if X_n is centered, we get by the strong law of large numbers that $C_n \to S_n + \sigma^2 I$, in probability, which coincides with our result on m_F as $c \downarrow 0$.

Many of the results that were proved in Silverstein and Choi [9] for the limiting spectral distribution of matrices of the form $\frac{1}{N}X_n^* T_n X_n$, with T_n $n \times n$ Hermitian, will be shown to hold for F, although the methods used here differ at times from the ones used in that paper. Two theorems from Silverstein and Choi [9] that will be needed are the following.

Theorem 1.1 [Theorem 2.1 of Silverstein and Choi [9]]. Let F be a p.d.f. and $x \in \mathbb{R}$. Suppose $\mathcal{I}m\, m_F(x) \equiv \lim_{z \in \mathbb{C}^+ \to x} \mathcal{I}m\, m_F(z)$ exists. Then F is differentiable at x, and its derivative is $\frac{1}{\pi}\mathcal{I}m\, m_F(x)$.

Theorem 1.2 [Theorem 2.2 of Silverstein and Choi [9]]. Let X be an open and bounded subset of $\mathbb{R}^n$, let Y be an open and bounded subset of $\mathbb{R}^m$, and let $f : \overline{X} \to Y$ be a function, continuous on X. If, for all $x_0 \in \partial X$, $\lim_{x \in X \to x_0} f(x) = f(x_0)$, then f is continuous on all of $\overline{X}$.

Our analysis is organized into three sections following the introduction. In section 2 we show that F has a density away from zero, and the density is analytic where it is positive. Section 3 provides a procedure for determining the support of F, and section 4 contains an analysis of the behavior of the density near certain points on the boundary of its support. In particular, it is shown that near these boundary points the density is similar to a square root function. Finally, the last section contains an example with specific choices for H, c, and σ and a discussion of the detection problem in array signal processing. For the example given, the graph of the density is shown along with a histogram and scatterplot of eigenvalues resulting from a simulation of the matrix C_n.

For notational convenience we will often write equation (1.1) in terms of the variable $b = 1 + \sigma^2 cm$ in which case we have the equation

$$b = 1 + \sigma^2 c \int \frac{dH(t)}{\frac{t}{b} - bz + \sigma^2(1 - c)}. \tag{1.1'}$$

Therefore, when we say that such a b satisfies (1.1′), the meaning is understood to be that the corresponding variable m satisfies (1.1). At times we will also write $b_F = 1 + \sigma^2 c m_F$ to make a reference to the Stieltjes transform m_F.

2. Existence of a Density

In this section we establish the following result.

THEOREM 2.1. For all $x \in \mathbb{R} - \{0\}$, $\lim_{z \in \mathbb{C}^+ \to x} m_F(z) \equiv \underline{m}(x)$ exists. The function $\underline{m}$ is continuous on $\mathbb{R} - \{0\}$ (Theorem 1.2), and F has a continuous derivative f on $\mathbb{R} - \{0\}$ given by $f(x) = \frac{1}{\pi} \mathcal{I}m\, \underline{m}(x)$ (Theorem 1.1). Furthermore, if $\mathcal{I}m\, \underline{m}(x) > 0$ ($f(x) > 0$) for $x \neq 0$, then $\underline{m}(x)$ is a solution to (1.1) for $z = x$, and the density f is analytic about x.

As indicated in the theorem, once existence of $\underline{m}$ is verified, we immediately have continuity of $\underline{m}$ and existence of the density f by Theorems 1.1 and 1.2. To prove the existence of $\underline{m}$ and the analyticity of f, we rely on a series of lemmas which will be stated and proved throughout this section.

We begin our analysis by establishing some useful definitions and inequalities that were originally presented in section 4 of Dozier and Silverstein [3].

Let $z = z_1 + iz_2 \in \mathbb{C}^+$, and let $m = m_F(z)$ and $b = b_1 + ib_2 = 1 + \sigma^2 cm$. Define the functions $g(b)$ and $G(b)$ by

$$g(b) = \int \frac{\sigma^2 c \frac{t}{|b|^2} dH(t)}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|^2}$$

$$G(b) = \int \frac{\sigma^2 c dH(t)}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|^2}.$$

Note that $G(b) > 0$, and since $H \neq 1_{[0,\infty)}$, we have $g(b) > 0$. Using these functions, we get from (1.1′) the following two equations

$$b_1 = 1 + b_1 g(b) + (\sigma^2(1-c) - \mathcal{R}e\, bz)G(b) \tag{2.1}$$

$$b_2 = b_2 g(b) + (\mathcal{I}m\, bz)G(b). \tag{2.2}$$

Writing $\mathcal{I}m\, bz = b_1 z_2 + b_2 z_1$, (2.2) implies

$$b_1 = b_2 \frac{1 - g(b) - z_1 G(b)}{z_2 G(b)}. \tag{2.3}$$

Since (2.1) can be written as

$$b_1(1 - g(b) + z_1 G(b)) = 1 + \sigma^2(1-c)G(b) + b_2 z_2 G(b)$$

we replace b_1 using (2.3) and get

$$b_2((1 - g(b))^2 - |z|^2 G^2(b)) = (1 + \sigma^2(1-c)G(b)) z_2 G(b) > 0$$

(recall $c \leq 1$).
Therefore,

$$(1 - g(b))^2 - |z|^2 G^2(b) > 0. \tag{2.4}$$

Since $G(b) > 0$ and $\mathcal{I}m\, bz = z_2 + \sigma^2 c \mathcal{I}m\, mz > 0$, we have by, (2.2), that $g(b) < 1$ and hence (2.4) implies

$$0 < |z| G(b) < 1 - g(b). \tag{2.5}$$

We now prove the following lemma.

LEMMA 2.1. Let $z = z_1 + iz_2 \in \mathbb{C}^+$, $m = m_F(z)$, and $b = b_1 + ib_2 = 1 + \sigma^2 cm$. Then we have the following three results:

(a) $b_1 > 0$,

(b) $|m| < \left(\frac{1}{\sigma^2 c |z|} \right)^{\frac{1}{2}}$,

(c) If $\lim_{z_n \to x} b \equiv \underline{b} = \underline{b}_1 + i\underline{b}_2$ exists for $\{z_n\} \subset \mathbb{C}^+$ and $x \in \mathbb{R} - \{0\}$, then $\underline{b}_1 > 0$.

Proof. For simplicity of notation we suppress the subscript n in the proof of (c). First, to prove (a), suppose $1 - g(b) - z_1 G(b) \leq 0$. Since $g(b) < 1$ we get

$$0 < (1 - g(b))^2 \leq z_1^2 G^2(b) < |z|^2 G^2(b),$$

a contradiction of (2.5). Therefore $1 - g(b) - z_1 G(b) > 0$, and since $b_2 > 0$, $z_2 > 0$, and $G(b) > 0$ we have $b_1 > 0$ by (2.3).

To prove (b) we first note that since $0 < g(b) < 1$, (2.5) gives

$$0 < G(b) < \frac{1}{|z|}. \tag{2.6}$$

Then using the Cauchy-Schwarz inequality we get

$$|m| \leq \int \frac{dH(t)}{\left| \frac{t}{b} - bz + \sigma^2(1-c) \right|} \leq \left(\int \frac{dH(t)}{\left| \frac{t}{b} - bz + \sigma^2(1-c) \right|^2} \right)^{\frac{1}{2}}$$

$$= \left(\frac{G(b)}{\sigma^2 c} \right)^{\frac{1}{2}} < \left(\frac{1}{\sigma^2 c |z|} \right)^{\frac{1}{2}}$$

Finally, for part (c) we note that part (b) gives $|\underline{b}| < \infty$. Solving (2.1) and (2.2) for $G(b)$ we find

$$G(b) = \frac{b_2}{\mathcal{I}m\,(b^2 z - b\sigma^2(1-c))} = \frac{1}{\mathcal{R}e\, bz + b_1 \frac{\mathcal{I}m\, bz}{b_2} - \sigma^2(1-c)}$$

$$= \frac{1}{2b_1 z_1 - b_2 z_2 + b_1^2 \frac{z_2}{b_2} - \sigma^2(1-c)}. \tag{2.7}$$

Since F is proper we have $\frac{z_2}{b_2} = \left(\sigma^2 c \int \frac{dF(\lambda)}{|\lambda - z|^2} \right)^{-1}$ is bounded as $z \to x$. Then if $\underline{b}_1 = 0$ and $c < 1$ we get

$$\lim_{z \to x} G(b) \equiv G = \frac{1}{-\sigma^2(1-c)} < 0,$$

a contradiction since (2.6) gives

$$0 \leq G \leq \frac{1}{|x|}. \tag{2.8}$$

If $c = 1$, then, as $z \to x$, $G(b)$ goes unbounded, again contradicting (2.8). Therefore, $\underline{b}_1 > 0$ and the proof is complete.

In the next lemma we will show that $m_F(z)$ has a unique limit as $z \to x \in \mathbb{R} - \{0\}$.

LEMMA 2.2. Let $\{z_n\}, \{\hat{z}_n\} \subset \mathbb{C}^+$ with z_n and $\hat{z}_n$ both converging to $x \in \mathbb{R} - \{0\}$ as $n \to \infty$. If $m = m_F(z_n) \to \underline{m}$ and $\hat{m} = m_F(\hat{z}_n) \to \underline{\hat{m}}$ as $n \to \infty$, then $\underline{m} = \underline{\hat{m}}$.

Proof. The result is obvious for $x < 0$ since m_F is analytic outside the support of F. Therefore, we assume $x > 0$. We let $b = b_1 + ib_2 = 1 + \sigma^2 cm$ and $\hat{b} = \hat{b}_1 + i\hat{b}_2 = 1 + \sigma^2 c\hat{m}$ and define the functions $g(\hat{b})$ and $G(\hat{b})$ in the same way that $g(b)$ and $G(b)$ are defined with the exception that b and z are replaced by $\hat{b}$ and $\hat{z}$, respectively.

To prevent the confusion of multiple subscripts, we will suppress the dependence on n of the sequence terms z_n, $\hat{z}_n$ and write $z_n = z = z_1 + iz_2$ and $\hat{z}_n = \hat{z} = \hat{z}_1 + i\hat{z}_2$.

We now take the difference $m - \hat{m} = \dfrac{(z-\hat{z})\beta_n}{1-\alpha_n}$ where

$$\alpha_n = \sigma^2 c \int \frac{\frac{t}{b\hat{b}} + z}{\left(\frac{t}{b} - bz + \sigma^2(1-c)\right)\left(\frac{t}{\hat{b}} - \hat{b}\hat{z} + \sigma^2(1-c)\right)} dH(t)$$

and

$$\beta_n = \int \frac{\hat{b}dH(t)}{\left(\frac{t}{b} - bz + \sigma^2(1-c)\right)\left(\frac{t}{\hat{b}} - \hat{b}\hat{z} + \sigma^2(1-c)\right)}.$$

Using the Cauchy-Schwarz inequality, (2.6), and Lemma 2.1 (b), we get for all n

$$|\beta_n| \le \frac{|\hat{b}|(G(b)G(\hat{b}))^{\frac{1}{2}}}{\sigma^2 c} \le \frac{1 + \sigma^2 c|\hat{m}|}{\sigma^2 c(|z||\hat{z}|)^{\frac{1}{2}}} \le \frac{1 + \left(\frac{\sigma^2 c}{|\hat{z}|}\right)^{\frac{1}{2}}}{\sigma^2 c\,(|z||\hat{z}|)^{\frac{1}{2}}} \le K < \infty.$$

Therefore $|m - \hat{m}| \le \dfrac{K|z-\hat{z}|}{|1-|\alpha_n||}$, and consequently we need only show that $|\alpha_n|$ stays uniformly away from 1.

Following the procedure from section 4 of Dozier and Silverstein [3], we use the triangle inequality followed by the Cauchy-Schwarz inequality to get

$$|\alpha_n| \le (g(b))^{\frac{1}{2}}(g(\hat{b}))^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}}. \tag{2.9}$$

Therefore, since $g(b)$, $g(\hat{b})$, $G(b)$, and $G(\hat{b})$ are bounded, we can choose a subsequence $\{n_j\}$ for which α_{n_j}, $g(b)$, $g(\hat{b})$, $G(b)$, and $G(\hat{b})$ converge, and we define their respective limits as α, g, $\hat{g}$, G, and $\hat{G}$.

For real numbers u and v with $u, v \in [0,1]$, it is easy to show that

$$(1-u)^{\frac{1}{2}}(1-v)^{\frac{1}{2}} \le 1 - (uv)^{\frac{1}{2}}, \tag{2.10}$$

with equality holding if and only if $u = v$.

Taking the limit in (2.9) we get

$$|\alpha| \leq g^{\frac{1}{2}}\hat{g}^{\frac{1}{2}} + |x|(G\hat{G})^{\frac{1}{2}}. \tag{2.11}$$

Let $\underline{b} = \underline{b}_1 + i\underline{b}_2 = 1 + \sigma^2 c\underline{m}$. From (2.9) and (2.5) we get for all j

$$|\alpha_{n_j}| < (1 - |z|G(b))^{\frac{1}{2}}(1 - |\hat{z}|G(\hat{b}))^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}}, \tag{2.12}$$

and then taking the limit we have

$$|\alpha| \leq (1 - xG)^{\frac{1}{2}}(1 - x\hat{G})^{\frac{1}{2}} + x(G\hat{G})^{\frac{1}{2}}. \tag{2.13}$$

If $G \neq \hat{G}$, then applying (2.10) to (2.13) we get the strict inequality

$$|\alpha| < 1 - (xGx\hat{G})^{\frac{1}{2}} + x(G\hat{G})^{\frac{1}{2}} = 1$$

as desired.

For the rest of the proof, we assume that $G = \hat{G}$.

From (2.7) we see that $G(b)$ converges if and only if $\frac{z_2}{b_2}$ converges. Therefore, $\frac{z_2}{b_2}$ and similarly $\frac{\hat{z}_2}{\hat{b}_2}$ must converge, and we call their respective limits y and $\hat{y}$.

Solving (2.2) for $g(b)$ gives

$$g(b) = 1 - \frac{\mathcal{I}m\, bz}{b_2}G(b) = 1 - \left(b_1\frac{z_2}{b_2} + z_1\right)G(b).$$

We solve for $g(\hat{b})$ the same way and substitute the results into (2.9) to get

$$|\alpha_{n_j}| \leq \left(1 - \left(b_1\frac{z_2}{b_2} + z_1\right)G(b)\right)^{\frac{1}{2}}\left(1 - \left(\hat{b}_1\frac{\hat{z}_2}{\hat{b}_2} + \hat{z}_1\right)G(\hat{b})\right)^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}}.$$

We then take the limit and use (2.10) to get

$$\begin{aligned} |\alpha| &\leq (1 - (\underline{b}_1 y + x)G)^{\frac{1}{2}}(1 - (\hat{\underline{b}}_1\hat{y} + x)G)^{\frac{1}{2}} + xG \\ &\leq 1 - \left((\underline{b}_1 y + x)^{\frac{1}{2}}(\hat{\underline{b}}_1\hat{y} + x)^{\frac{1}{2}} - x\right)G. \end{aligned} \tag{2.14}$$

By Lemma 2.1 (c), we have $\underline{b}_1 > 0$ and $\hat{\underline{b}}_1 > 0$. Therefore if either $y > 0$ or $\hat{y} > 0$, we have $(\underline{b}_1 y + x)^{\frac{1}{2}}(\hat{\underline{b}}_1\hat{y} + x)^{\frac{1}{2}} > x$, and hence $|\alpha| < 1$ by (2.14).

Suppose $y = \hat{y} = 0$. Then (2.7) gives

$$\frac{1}{2\underline{b}_1 x - \sigma^2(1-c)} = G = \hat{G} = \frac{1}{2\hat{\underline{b}}_1 x - \sigma^2(1-c)},$$

and hence $\underline{b}_1 = \hat{\underline{b}}_1$.

If $\underline{b}_2 = \hat{\underline{b}}_2 = 0$, we are done. Suppose that either $\underline{b}_2 > 0$ or $\hat{\underline{b}}_2 > 0$. Define

$$k_{n_j}(t) \equiv \left|\frac{t}{b\hat{b}}\right| + |z| - \left|\frac{t}{b\hat{b}} + z\right| \qquad \text{for} \quad t \geq 0.$$

Since $\mathcal{I}m \, \frac{1}{b\hat{b}} = \frac{-(b_1\hat{b}_2 + b_2\hat{b}_1)}{|b\hat{b}|^2} \to \frac{-(\underline{b}_1\hat{\underline{b}}_2 + \underline{b}_2\hat{\underline{b}}_1)}{|\underline{b}\hat{\underline{b}}|^2} < 0$ as $j \to \infty$, then z and $\frac{1}{b\hat{b}}$ are noncolinear for j large. Therefore, since $k_{n_j}(t)$ is the residual of the triangle inequality, we have for large j, $k_{n_j}(t) \geq 0$ for $t \geq 0$ with $k_{n_j}(t) = 0$ if and only if $t = 0$.

Define

$$\gamma_{n_j} \equiv \sigma^2 c \int \frac{k_{n_j}(t) dH(t)}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|\left|\frac{t}{\hat{b}} - \hat{b}\hat{z} + \sigma^2(1-c)\right|}.$$

Since $k_{n_j}(t) \leq \left|\frac{t}{b\hat{b}}\right| + |z|$, we have

$$\gamma_{n_j} \leq \sigma^2 c \int \frac{\left|\frac{t}{b\hat{b}}\right| + |z|}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|\left|\frac{t}{\hat{b}} - \hat{b}\hat{z} + \sigma^2(1-c)\right|} dH(t)$$

$$\leq (g(b))^{\frac{1}{2}}(g(\hat{b}))^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}} \leq 1$$

for all j. Therefore by Fatou's lemma we get

$$\gamma \equiv \sigma^2 c \int \frac{\lim_{j\to\infty} k_{n_j}(t)\, dH(t)}{\left|\frac{t}{\underline{b}} - \underline{b}x + \sigma^2(1-c)\right|\left|\frac{t}{\hat{\underline{b}}} - \hat{\underline{b}}x + \sigma^2(1-c)\right|} \leq \liminf_{j\to\infty} \gamma_{n_j} \leq 1.$$

Since H is proper, $H \neq 1_{[0,\infty)}$, and $\underline{b}$, $\hat{\underline{b}}$ are finite we get $\gamma > 0$.

Going back to the definition of α we follow similar steps as before to derive

$$|\alpha_{n_j}| \leq \sigma^2 c \int \frac{\left|\frac{t}{b\hat{b}}\right| + |z| - k_{n_j}(t)}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|\left|\frac{t}{\hat{b}} - \hat{b}\hat{z} + \sigma^2(1-c)\right|} dH(t)$$

$$\leq (g(b))^{\frac{1}{2}}(g(\hat{b}))^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}} - \gamma_{n_j}$$

$$< (1 - |z|G(b))^{\frac{1}{2}}(1 - |\hat{z}|G(\hat{b}))^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}} - \gamma_{n_j}.$$

Then

$$|\alpha| \leq \liminf_{j\to\infty} \left((1 - |z|G(b))^{\frac{1}{2}}(1 - |\hat{z}|G(\hat{b}))^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}} - \gamma_{n_j}\right)$$

$$\leq \liminf_{j\to\infty} \left((1 - |z|G(b))^{\frac{1}{2}}(1 - |\hat{z}|G(\hat{b}))^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}}\right) - \liminf_{j\to\infty} \gamma_{n_j}$$

$$= 1 - \liminf_{j\to\infty} \gamma_{n_j}$$

$$\leq 1 - \gamma < 1.$$

Therefore in every case we have $\underline{m} = \hat{\underline{m}}$, and hence the proof is complete.

By Theorems 1.1 and 1.2 and Lemmas 2.1 (b) and 2.2 we now have the existence and continuity of both $\underline{m}$ and f on $\mathbb{R} - \{0\}$. Moreover, when $f(x) > 0$ we have

$$\mathcal{I}m \left(\frac{t}{\underline{b}(x)} - \underline{b}(x)x - \sigma^2(1-c)\right) \leq -\mathcal{I}m\, \underline{b}(x)x < 0$$

for all $t \geq 0$, and therefore, by dominated convergence, $\underline{m}(x)$ satisfies (1.1) for $z = x$. Therefore, the only part of Theorem 2.1 that remains to be shown is the analyticity of f.

The following lemma presents a slightly stronger result on uniqueness of solutions to (1.1) than was stated in Theorem 4.1 of Dozier and Silverstein [3].

LEMMA 2.3. Let $m = m_1 + im_2 \in \mathbb{C}^+$ and $b = b_1 + ib_2 = 1 + \sigma^2 cm$. If, for $z = z_1 + iz_2 \in \mathbb{C}^+$, m is a solution to equation (1.1) and $\mathcal{I}m\, bz > 0$, then m is unique.

Proof. The difference between this lemma and Theorem 4.1 of Dozier and Silverstein [3] is that here we assume $\mathcal{I}m\, bz > 0$ instead of $\mathcal{I}m\, mz \geq 0$. The proof, however, is exactly the same for both cases since the theorem's proof only uses the inequality $\mathcal{I}m\, mz \geq 0$ to establish that $\mathcal{I}m\, bz > 0$ by the expression $\mathcal{I}m\, bz = z_2 + \sigma^2 c \mathcal{I}m\, mz > 0$. Hence, the proof is complete.

We now complete the proof of Theorem 2.1 with the following lemma.

LEMMA 2.4. If $x_0 \in (0, \infty)$ and $f(x_0) > 0$, then f is analytic near x_0.

Proof. Let $\underline{b} = \underline{b}_1 + i\underline{b}_2 = 1 + \sigma^2 c\underline{m}(x_0)$. For $z \in \mathbb{C}^+$ and any $m \in \mathbb{C}^+$ satisfying (1.1), we get

$$\frac{m}{1 + \sigma^2 cm} = \int \frac{dH(t)}{t - (b^2 z - b\sigma^2(1-c))} = m_H(b^2 z - b\sigma^2(1-c)), \tag{2.15}$$

where $m_H(\cdot)$ denotes the Stieltjes transform of H and $b = 1 + \sigma^2 cm$. Let $\underline{w} \equiv \underline{b}^2 x_0 - \underline{b}\sigma^2(1-c)$. Since the denominator in (2.7) is bounded, we can tighten inequality (2.8) to get

$$0 < G \leq \frac{1}{x_0}. \tag{2.16}$$

From (2.7) we get $\mathcal{I}m\, \underline{w} = \frac{\underline{b}_2}{G}$, and since $\underline{b}_2 = \pi\sigma^2 cf(x_0) > 0$ we have $\mathcal{I}m\, \underline{w} > 0$, and hence m_H is analytic near $\underline{w}$.

First, suppose that $m_H'(\underline{w}) \neq 0$. Then in a neighborhood of $\underline{w}$, the analytic inverse m_H^{-1} exists. It is clear that for z near x_0 and b near $\underline{b}$, we have $w \equiv b^2 z - b\sigma^2(1-c)$ near $\underline{w}$. Therefore, if b is near $\underline{b}$ and b satisfies equation (1.1′) for z near x_0, then we have

$$\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right) = m_H(b^2 z - b\sigma^2(1-c)) = m_H(w), \tag{2.17}$$

and hence

$$z = \frac{1}{b^2} m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right) + \frac{1}{b}\sigma^2(1-c). \tag{2.18}$$

Let $z(b)$ be the right hand side of (2.18). In a neighborhood of $\underline{b}$, $z(b)$ is clearly analytic, and we will show that it is also one-to-one.

For complex numbers b, $\hat{b}$ near $\underline{b}$ and z close enough to x_0 so that $\mathcal{I}m\, bz > 0$ and $\mathcal{I}m\, \hat{b}z > 0$, define the function

$$\alpha = \sigma^2 c \int \frac{\frac{t}{b\hat{b}} + z}{\left(\frac{t}{b} - bz + \sigma^2(1-c)\right)\left(\frac{t}{\hat{b}} - \hat{b}z + \sigma^2(1-c)\right)} dH(t).$$

Note that for $t \geq 0$,

$$\mathcal{I}m\left(\frac{t}{b} - bz + \sigma^2(1-c)\right) = -\frac{t\mathcal{I}m\, b}{|b|^2} - \mathcal{I}m\, bz < 0,$$

and similarly for $\hat{b}$. Therefore the integrand of α is bounded since for any $t \geq 0$

$$\frac{\left|\frac{t}{b\hat{b}} + z\right|}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|\left|\frac{t}{\hat{b}} - \hat{b}z + \sigma^2(1-c)\right|} \leq \frac{\frac{t}{|b\hat{b}|} + |z|}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|\mathcal{I}m\,\hat{b}z}$$

$$\leq \frac{1}{|\hat{b}|\mathcal{I}m\,\hat{b}z}\left|\frac{\frac{t}{b}}{\frac{t}{b} - bz + \sigma^2(1-c)}\right| + \frac{|z|}{(\mathcal{I}m\, bz)(\mathcal{I}m\,\hat{b}z)}$$

$$= \frac{1}{|\hat{b}|\mathcal{I}m\,\hat{b}z}\left|1 - \frac{-bz + \sigma^2(1-c)}{\frac{t}{b} - bz + \sigma^2(1-c)}\right| + \frac{|z|}{(\mathcal{I}m\, bz)(\mathcal{I}m\,\hat{b}z)}$$

$$\leq \frac{1}{|\hat{b}|\mathcal{I}m\,\hat{b}z}\left(1 + \frac{|b||z| + \sigma^2(1-c)}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|}\right) + \frac{|z|}{(\mathcal{I}m\, bz)(\mathcal{I}m\,\hat{b}z)}$$

$$\leq \frac{1}{|\hat{b}|\mathcal{I}m\,\hat{b}z}\left(1 + \frac{|b||z| + \sigma^2(1-c)}{\mathcal{I}m\, bz}\right) + \frac{|z|}{(\mathcal{I}m\, bz)(\mathcal{I}m\,\hat{b}z)} < K,$$

and hence α is well-defined and, in fact, continuous in the variables b, $\hat{b}$, and z. Define $\underline{\alpha}$ to be the value of α when $b = \hat{b} = \underline{b}$ and $z = x_0$, that is,

$$\underline{\alpha} = \sigma^2 c \int \frac{\frac{t}{\underline{b}^2} + x_0}{\left(\frac{t}{\underline{b}} - \underline{b}x_0 + \sigma^2(1-c)\right)^2} dH(t).$$

Define

$$\underline{k}(t) \equiv \left|\frac{t}{\underline{b}^2}\right| + x_0 - \left|\frac{t}{\underline{b}^2} + x_0\right|$$

and

$$\underline{\gamma} \equiv \sigma^2 c \int \frac{\underline{k}(t)}{\left|\frac{t}{\underline{b}} - \underline{b}x_0 + \sigma^2(1-c)\right|^2} dH(t).$$

Now, $\mathcal{I}m\,\frac{1}{\underline{b}^2} = -\frac{b_1 b_2}{|\underline{b}|^2} < 0$, and therefore $\frac{1}{\underline{b}^2}$ and x_0 are noncolinear. Since $\underline{k}$ is the residual of the triangle inequality, we have $\underline{k}(t) \geq 0$ for $t \geq 0$ with $\underline{k}(t) = 0$ if and only if $t = 0$. Therefore since $H \neq 1_{[0,\infty)}$, we have $\underline{\gamma} > 0$, and since (2.2) gives $1 - g = x_0 G$, we get, as in the proof of Lemma 2.2,

$$|\underline{\alpha}| \leq \sigma^2 c \int \frac{\left|\frac{t}{\underline{b}^2} + x_0\right|}{\left|\frac{t}{\underline{b}} - \underline{b}x_0 + \sigma^2(1-c)\right|^2} dH(t) = \sigma^2 c \int \frac{\frac{t}{|\underline{b}|^2} + x_0 - \underline{k}(t)}{\left|\frac{t}{\underline{b}} - \underline{b}x_0 + \sigma^2(1-c)\right|^2} dH(t)$$

$$= g + x_0 G - \underline{\gamma} = 1 - \underline{\gamma} < 1.$$

Suppose we have b and $\hat{b}$ both satisfying (1.1′) for the same z, where b, $\hat{b}$ are close to $\underline{b}$ and z is close enough to x_0 so that $\mathcal{I}m\, bz > 0$ and $\mathcal{I}m\, \hat{b}z > 0$. Then we can write $b - \hat{b} = (b - \hat{b})\alpha$, and since α is continuous for the variables b, $\hat{b}$, z and $|\underline{\alpha}| < 1$, we must have $|\alpha| < 1$ for all of these b, $\hat{b}$ and z sufficiently close to $\underline{b}$ and x_0, respectively. Therefore, $b = \hat{b}$. Then the function $z(b)$ is one-to-one near $\underline{b}$ and hence has an analytic inverse $b(z)$ for z near x_0. By Lemma 2.3 we must have $b(z) = 1 + \sigma^2 c m_F(z)$ for $z \in \mathbb{C}^+$, and hence m_F extends analytically onto an interval about x_0. Therefore we get

$$\underline{m}(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for x near x_0 and some $a_n \in \mathbb{C}$, and hence

$$f(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} \mathcal{I}m\, a_n (x - x_0)^n. \tag{2.19}$$

Now, suppose $m_H'(\underline{w}) = 0$. We form the function u of the two complex variables b, z by

$$u(b, z) = m_H(b^2 z - b\sigma^2(1 - c)) - \frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)$$

which is analytic near $(\underline{b}, x_0) \in \mathbb{C}^2$. Then we have $u(\underline{b}, x_0) = 0$. Taking the derivative with respect to b we get

$$\frac{\partial u}{\partial b}(\underline{b}, x_0) = m_H'(\underline{w})(2\underline{b}x_0 - \sigma^2(1 - c)) - \frac{1}{\sigma^2 c \underline{b}^2} = -\frac{1}{\sigma^2 c \underline{b}^2} \neq 0.$$

Then by the implicit function theorem (Krantz [4] p.54) there is a unique analytic solution $b(z)$ in a neighborhood of x_0 such that $b(x_0) = \underline{b}$. Since m_F is an analytic solution to (1.1) in $\mathbb{C}^+$, we must have $b(z) = 1 + \sigma^2 c m_F(z)$ by uniqueness of $b(z)$, and hence m_F extends analytically to an interval about x_0, and again we have (2.19). Therefore, $f(x)$ is analytic where it is positive, and the proof is complete.

3. The Support of F

In this section we present results on the support of the limiting distribution F. Let S_F and S_H denote the support of F and H, respectively. Clearly, by definition of F and H, we have $S_F \subset [0, \infty)$ and $S_H \subset [0, \infty)$. We begin our analysis of S_F with the following result.

THEOREM 3.1. F has no mass at 0.

Proof. The method we will use to prove the lemma was previously used in Silverstein and Choi [9].

For any p.d.f. G we have

$$\lim_{y \downarrow 0} iy m_G(iy) = -G\{0\} + \lim_{y \downarrow 0} \int_{(0,\infty)} \frac{iy}{\lambda - iy} dG(\lambda) = -G\{0\},$$

by dominated convergence, and therefore, if $G\{0\} > 0$, we must have $|m_G(iy)| \to \infty$ as $y \downarrow 0$.

Suppose $F\{0\} > 0$. From (1.1) we have

$$iym(iy) = \int \frac{iy}{\frac{t}{1+\sigma^2 cm(iy)} - (1+\sigma^2 cm(iy))iy + \sigma^2(1-c)} dH(t).$$

Since $F\{0\} > 0$ we have, for any $t \geq 0$, as $y \downarrow 0$

$$\frac{iy}{\frac{t}{1+\sigma^2 cm(iy)} - (1+\sigma^2 cm(iy))iy + \sigma^2(1-c)} \to \frac{0}{\sigma^2 cF\{0\} + \sigma^2(1-c)} = 0,$$

and since

$$\left| \frac{iy}{\frac{t}{1+\sigma^2 cm(iy)} - (1+\sigma^2 cm(iy))iy + \sigma^2(1-c)} \right| \leq \frac{y}{\mathcal{I}m\, iyb(iy)} = \frac{1}{b_1(iy)} < \infty,$$

by Lemma 2.1 (a), we have, by dominated convergence, $\lim_{y \downarrow 0} iym(iy) = 0$, a contradiction. Therefore, $F\{0\} = 0$.

The fact that $F\{0\} = 0$ gives no information on whether or not $0 \in S_F$. Simulations have shown that either case can occur, depending on H and the values of c and σ.

A method to identify the support of F is presented next.

First, we give a lemma that will be used in the proof of Theorem 3.3.

LEMMA 3.1. If b,$\mathbf{b} \in \mathbb{R}$ are positive and both satisfy $(1.1')$ for $z = x \in \mathbb{R}$, $x < 0$, then $b = \mathbf{b}$.

Proof. First, note that for $t \geq 0$, $b > 0$, and $x < 0$ we have

$$\frac{1}{|\frac{t}{b} - bx + \sigma^2(1-c)|} = \frac{1}{\frac{t}{b} + b|x| + \sigma^2(1-c)} \leq \frac{1}{b|x|} < \infty,$$

and therefore the integral in $(1.1')$ is well-defined for both b and $\mathbf{b}$. We write $b - \mathbf{b} = (b - \mathbf{b})\alpha$, where

$$\alpha = \sigma^2 c \int \frac{\frac{t}{b\mathbf{b}} + x}{(\frac{t}{b} - bx + \sigma^2(1-c))(\frac{t}{\mathbf{b}} - \mathbf{b}x + \sigma^2(1-c))} dH(t).$$

Again, following the procedure from section four of Dozier and Silverstein [3], we use the Cauchy-Schwarz and triangle inequalities to get

$$|\alpha| \leq (g(b))^{\frac{1}{2}} (g(\mathbf{b}))^{\frac{1}{2}} + |x|(G(b)G(\mathbf{b}))^{\frac{1}{2}}. \tag{3.1}$$

From (2.1) we get $b(1 - g(b) + xG(b)) = 1 + \sigma^2(1-c)G(b) > 0$, and since $b > 0$ we get

$$g(b) < 1 + xG(b) = 1 - |x|G(b) \tag{3.2}$$

and similarly for $\mathbf{b}$. Substituting this into (3.1) and using (2.10) we get

$$|\alpha| < (1 - |x|G(b))^{\frac{1}{2}} (1 - |x|G(\mathbf{b}))^{\frac{1}{2}} + |x|(G(b)G(\mathbf{b}))^{\frac{1}{2}}$$

$$\leq 1 - (|x|G(b)|x|G(\mathbf{b}))^{\frac{1}{2}} + |x|(G(b)G(\mathbf{b}))^{\frac{1}{2}} = 1.$$

Therefore, $b = \mathbf{b}$, and the proof is complete.

Suppose we have $x \in \mathbb{R} - \{0\}$. If $x \in S_F^c$, we have that $\underline{m}(x)$ is real, continuous, and increasing, and therefore so is $\underline{m}^{-1}(x)$. Let $b(z) = b_1(z) + ib_2(z) = 1 + \sigma^2 c m_F(z)$. Since $b(z)$ is a well-defined, analytic function for z in a neighborhood of x, we have that the function $w(z) \equiv b^2(z)z - b(z)\sigma^2(1-c)$ is also well-defined and analytic in such a neighborhood. In the next theorem, we will show that $w(x) \in S_H^c$, and therefore we may write the inverse of $\underline{m}$, expressed in terms of $b \in \mathbb{R}$, as

$$x(b) = \frac{1}{b^2} m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right) + \frac{1}{b}\sigma^2(1-c).$$

THEOREM 3.2. If $x \in S_F^c$, then $w(x) \in S_H^c$.

Proof. Let $(l_1, l_2) \subset [L_1, L_2] \subset S_H^c$ and choose $x_0 \in (l_1, l_2)$. Since $x_0 \in S_F^c$, $m_F(z)$ is analytic in a neighborhood V of x_0 with $V \cap \mathbb{R} \subset (l_1, l_2)$, and therefore $w(z)$ is also analytic in V. Note that $w(z) = z + 2\sigma^2 c m_F(z) z + (\sigma^2 c)^2 m_F^2(z) z - \sigma^2(1-c) - \sigma^2 c \sigma^2 (1-c) m_F(z)$. Let $\zeta = u + iv$, where $u \in \mathbb{R}$ is fixed and $v \to \infty$. Since $m_F(\zeta) \to 0$, $m_F(\zeta)\zeta$ is bounded, and $m_F^2(\zeta)\zeta \to 0$, we have $w(\zeta) \to \infty$, and hence $w(z)$ is nonconstant. Therefore, by the open mapping theorem, $w(V)$ is an open set.

For $z \in \mathbb{C}^+$ we have $b(z) \in \mathbb{C}^+$, and therefore $w(z) \in \mathbb{C}^+$ by (2.7). Therefore, by (1.1) and Lemma 2.1 (a), we get for any $z \in \mathbb{C}^+$

$$\frac{m_F(z)}{1 + \sigma^2 c m_F(z)} = m_H(w(z)) = \int \frac{dH(t)}{t - w(z)},$$

which gives

$$\mathcal{I}m\, m_H(w(z)) = \mathcal{I}m\, w(z) \int \frac{dH(t)}{|t - w(z)|^2} = \frac{\mathcal{I}m\, m_F(z)}{|1 + \sigma^2 c m_F(z)|^2}. \tag{3.3}$$

Let $w_0 \in w(V) \cap \mathbb{R}$ be arbitrary. Take a sequence $\{w_j\} \subset w(V) \cap \mathbb{C}^+$ such that $w_j \to w_0$. There exists a sequence $\{z_j\} \subset V$ for which $w_j = w(z_j)$ for each j. For any $z \in V$ we have $\overline{b(\overline{z})} = b(z)$, and consequently, $\overline{w(\overline{z})} = w(z)$. Therefore, $\{z_j\} \subset \mathbb{C}^+$. Since the z_j's are bounded, there exists a subsequence $\{z_{j_k}\} \subset \{z_j\}$ that converges to some $z_0 \in \overline{V}$. If $z_0 \in \mathbb{C}^+$, then $G(b(z_0)) > 0$ and $b_2(z_0) > 0$, and therefore (2.3) gives $\mathcal{I}m\, w(z_0) = \mathcal{I}m\, w_0 > 0$, a contradiction. Then we must have $z_0 \in \mathbb{R}$, and hence $z_0 \in S_F^c$. Therefore $\mathcal{I}m\, m_F(z_0) = 0$. If $z_0 \neq 0$, Lemma 2.1 (c) gives $b(z_0) > 0$. If $z_0 = 0$ we may assume, without loss of generality, that $0 \in [L_1, L_2]$, and therefore we have

$$b(z_0) = 1 + \sigma^2 c \int \frac{dF(\lambda)}{\lambda - z_0} = 1 + \sigma^2 c \int_{(L_2, \infty)} \frac{dF(\lambda)}{\lambda} > 1.$$

Therefore, in either case, $b(z_0) > 0$, and (3.3) gives

$$\lim_{k \to \infty} \mathcal{I}m\, m_H(w(z_{j_k})) = \frac{\mathcal{I}m\, m_F(z_0)}{|b(z_0)|^2} = 0.$$

Hence by Theorem 1.1, H is differentiable at w_0 and its derivative is 0. Since w_0 is arbitrary in $w(V) \cap \mathbb{R}$, we have $w(V) \cap \mathbb{R} \subset S_H^c$, and therefore $w(x_0) \in S_H^c$, and since x_0 is arbitrary in S_F^c, the proof is complete.

So far we have shown that if we have an x outside the support of F, the corresponding $w(x)$ is outside the support of H, and we have an expression for the inverse of $\underline{m}$. Therefore, if we graph the inverse $x(b)$ and identify an interval of points in S_F^c on the vertical axis, $x(b)$ will be increasing on that interval, but does the presence of an interval on the vertical axis for which $x(b)$ is increasing always yield an interval in S_F^c? The answer is yes, if $b > 0$, as Theorem 3.3 will show. To prove this semi-converse we proceed as follows.

Suppose we have $w_0 \in (l_1, l_2) \subset [L_1, L_2] \subset S_H^c$. Then $m_H(\cdot)$ is increasing on (l_1, l_2) and maps (l_1, l_2) onto some interval (d_1, d_2). Now, $\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)$ is an increasing function of b from $(0, \infty)$ onto $(-\infty, \frac{1}{\sigma^2 c})$. Since $b \leq 0$ does not correspond to our Stieltjes transform by Lemma 2.1 (a),(c) , we may assume w_0 is chosen so that $(d_1, d_2) \subset (-\infty, \frac{1}{\sigma^2 c})$. Therefore there is an interval $(k_1, k_2) \subset (0, \infty)$ such that the mapping

$$b \mapsto \frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right) \tag{3.4}$$

is a one-to-one correspondence from (k_1, k_2) to (d_1, d_2). Therefore $m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right)$ is well-defined from (k_1, k_2) to (l_1, l_2), and hence we define

$$x(b) = \frac{1}{b^2} m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right) + \frac{1}{b}\sigma^2(1 - c) \tag{3.5}$$

for $b \in (k_1, k_2)$. The next theorem will show that at a point $b \in (k_1, k_2)$ for which $x'(b) > 0$, we have $x(b) \in S_F^c$, and $b = 1 + \sigma^2 c m_F(x(b))$.

THEOREM 3.3. Let $b \in (k_1, k_2)$ and $x'(b) > 0$. Then $x(b) \in S_F^c$ and $b = 1 + \sigma^2 c m_F(x(b))$.

Proof. Let $(\underline{k}_1, \underline{k}_2) \subset (k_1, k_2)$ be an interval on which $x'(b) > 0$. Fix $\mathbf{b} \in (\underline{k}_1, \underline{k}_2)$. If $x(\mathbf{b}) < 0$, we immediately have $x(\mathbf{b}) \in S_F^c$, and by Lemma 3.1 we must have $\mathbf{b} = 1 + \sigma^2 c m_F(x(\mathbf{b}))$. Therefore we assume $x(\mathbf{b}) \geq 0$. Let D be an open set in $\mathbb{C}$ such that $D \cap \mathbb{R} = (\underline{k}_1, \underline{k}_2)$. Since x is analytic on $(\underline{k}_1, \underline{k}_2)$, we may write $x(b)$ in a power series expansion centered at $\mathbf{b}$, and therefore, for $b \in D$, the function

$$z(b) \equiv x(\mathbf{b}) + \sum_{j=1}^{\infty} \frac{x^{(j)}(\mathbf{b})}{j!}(b - \mathbf{b})^j \tag{3.6}$$

is the analytic extension of x onto D. Using (3.6) we write $z(b) = x(\mathbf{b}) + x'(\mathbf{b})(b - \mathbf{b}) + \theta(b)$ where $\theta(b) = o(b - \mathbf{b})$. Since $x'(\mathbf{b}) > 0$, it is clear that we may choose $\hat{b} = \hat{b}_1 + i\hat{b}_2 \in D \cap \mathbb{C}^+$ sufficiently close to $\mathbf{b}$ to ensure that $z(\hat{b}) \in \mathbb{C}^+$, and since $\hat{b}z(\hat{b}) = \hat{b}x(\mathbf{b}) + x'(\mathbf{b})\hat{b}(\hat{b} - \mathbf{b}) + \hat{b}\theta(\hat{b})$ and $x(\mathbf{b}) \geq 0$ we have $\mathcal{I}m\, \hat{b}z(\hat{b}) = \hat{b}_2(x(\mathbf{b}) + x'(\mathbf{b})(2\hat{b}_1 - \mathbf{b})) + \mathcal{I}m\, \hat{b}\theta(\hat{b}) > 0$ for $\hat{b} \in D \cap \mathbb{C}^+$ close enough to $\mathbf{b}$. Therefore we have $\mathcal{I}m\, z(\hat{b}) > 0$, $\mathcal{I}m\, \hat{b}z(\hat{b}) > 0$, and

$$z(\hat{b}) = \frac{1}{\hat{b}^2} m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{\hat{b}}\right)\right) + \frac{1}{\hat{b}}\sigma^2(1 - c). \tag{3.7}$$

Hence, by Lemma 2.3, $\hat{b}$ is the unique solution to (1.1′) for $z = z(\hat{b})$, that is, $\hat{b} = 1 + \sigma^2 c m_F(z(\hat{b}))$. Therefore, $b_F = 1 + \sigma^2 c m_F$ extends analytically onto a neighborhood B of $\mathbf{b}$ and its inverse is given by (3.7).

Choose a sequence $\{z_j\} \subset z(B) \cap \mathbb{C}^+$ such that $z_j \to z(\mathbf{b})$ $(= x(\mathbf{b}))$. Then we have $b_F(z_j) = 1 + \sigma^2 c m_F(z_j) \to b_F(z(\mathbf{b})) = \mathbf{b}$, and consequently $\mathcal{I}m\, m_F(z_j) \to 0$ as $j \to \infty$. By Theorem 1.1, F is differentiable at $x(\mathbf{b})$, and it's derivative is 0. Since $\mathbf{b} \in (\underline{k}_1, \underline{k}_2)$ is arbitrary we have $F'(x) = 0$ for all $x = x(b) \in (x(\underline{k}_1), x(\underline{k}_2))$, and therefore these x's are outside S_F. Moreover, m_F is analytic in $\mathbb{C}^+ \cup (x(\underline{k}_1), x(\underline{k}_2))$, and therefore $b = 1 + \sigma^2 c m_F(x(b))$ for any $b \in (k_1, k_2)$ for which $x'(b) > 0$, and this completes the proof.

As a result of Theorems 3.2 and 3.3, we now have a method whereby we may graphically identify the support of F. The first step of the procedure is to choose an open interval $I_H \subset S_H^c$ such that $\overline{I}_H$ is not in S_H^c, that is, I_H is not a subset of a larger interval in S_H^c. On I_H, m_H is increasing and maps to an interval (d_1, d_2). Since the function (3.4) maps positive values of b onto $(-\infty, \frac{1}{\sigma^2 c})$, we take only those intervals I_H for which $(\underline{d}_1, \underline{d}_2) \equiv (d_1, d_2) \cap (-\infty, \frac{1}{\sigma^2 c})$ is nonempty, and disregard any I_H for which this intersection is empty. Let (k_1, k_2) be the pre-image of (d_1, d_2) under the mapping given in (3.4). Therefore, $m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right)$ is well-defined from (k_1, k_2) to $\tilde{I}_H \equiv \{t \in I_H : m_H(t) \in (\underline{d}_1, \underline{d}_2)\}$, and hence we may graph the function $x(b)$ given by (3.5) for $b \in (k_1, k_2)$. We then identify all intervals on the vertical axis where the graph of x is increasing. By Theorem 3.3, we know that these intervals are outside S_F, and therefore we remove these intervals from $\mathbb{R}$, and S_F must be contained in what is left. We continue in this manner for every interval $I_H \subset S_H^c$. Let D be the resulting set. Since, by Theorem 3.2, every $x \in S_F^c$ corresponds (via $w(x)$) to a point in S_H^c, we must have $D = S_F$. Also, for each interval $I_F \subset S_F^c$, there is only one interval $I_H \subset S_H^c$ for which our procedure produces I_F. In other words, the intervals outside S_F that are being removed from $\mathbb{R}$ in the above procedure will not overlap each other. To see this, we note that (3.5) gives

$$m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right) = b^2 x(b) - b\sigma^2(1-c). \tag{3.8}$$

By Theorem 3.3, for each $x \in I_F$, there is a unique b, namely $b = 1 + \sigma^2 c m_F(x)$, such that $x = x(b)$. Therefore, I_F uniquely determines the range of the left-hand side of (3.8), which is an interval in S_H^c. Consequently, once we eliminate an interval from being in S_F, we will never again encounter any portion of this interval in subsequent steps of the procedure.

4. Behavior Near a Boundary Point

We now focus on the behavior of the density f near boundary points of S_F. Let a be a left end-point of S_F, and let $\epsilon > 0$ be sufficiently small so that $(a - \epsilon, a) \subset S_F^c$. Therefore, by the previous section, there exists an interval $(l_1, l_2) \subset S_H^c$ from which we can construct a well-defined, analytic function $x(b)$ given by the representation in (3.5), for b in some interval $(k_1, k_2) \subset (0, \infty)$, such that $(a - \epsilon, a)$ is in the range of $x(b)$ and $x'(b)$ is positive over these range values. We now assume that $[a - \epsilon, a]$ is in the range of $x(b)$, and, in particular, we define $b_*, b_a \in (k_1, k_2)$ so that $x(b_*) = a - \epsilon$ and $x(b_a) = a$. Therefore $b_* < b_a$, and $x(b)$ is defined on both sides of b_a.

Note that our assumption may not occur for certain choices of H. It may be the case that $\lim_{b \uparrow b_a} x(b)$ exists, but $x(b)$ is not defined at b_a, which can possibly occur if $b_a^2 a - b_a \sigma^2(1-c) \in \partial S_H$ and $m_H'(w)$ exists as $w \to b_a^2 a - b_a\sigma^2(1-c)$. However, our assumption is valid, for example, when H is discrete, since m'_H will not exist on ∂S_H. This constitutes the most relevant cases of application of our model. A non-discrete H would only be considered if it approximates the population eigenvalues in an analytically tractable manner.

Since $x(b)$ is analytic with $x'(b) > 0$ for all $b \in (b_*, b_a)$ and a is a left end-point of S_F, we must have $x'(b_a) = 0$, and the next theorem will imply that b_a is a relative maximum of x.

THEOREM 4.1. For some $\delta > 0$ for which $b_a + \delta < k_2$ we have $x'(b) < 0$ for all $b \in (b_a, b_a + \delta)$.

Proof. Suppose $x^{(j)}(b_a)$ is the first non-vanishing derivative of $x(b)$ at b_a. Then for all b in some interval $(b_a, b_a+\delta) \subset (b_a, k_2)$, $x^{(j)}(b)$ is of one sign, and therefore each of the first $j-1$ derivatives do not change sign in this interval. If $x'(b) > 0$ on $(b_a, b_a+\delta)$ then we would have $(x(b_a), x(b_a+\delta)) = (a, x(b_a+\delta)) \subset S_F^c$, and consequently, a would be an isolated point in S_F, an impossibility since F has a continuous density on $\mathbb{R} - \{0\}$. Therefore we must have $x'(b) < 0$ for all $b \in (b_a, b_a+\delta)$, and the proof is complete.

Let $k^* \in (b_a, k_2)$ be such that $x'(b) < 0$ for all $b \in (b_a, k^*)$. Define the interval $(\underline{l}_1, \underline{l}_2) \subset (l_1, l_2)$ to be the image of (b_*, k^*) under the mapping $m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right)$. For $z \in \mathbb{C}^+$ let $b_F(z) \equiv 1 + \sigma^2 c m_F(z)$. Write $\lim_{z \in \mathbb{C}^+ \to x} b_F(z) \equiv \underline{b}(x) = \underline{b}_1(x) + i\underline{b}_2(x)$ for $x \in \mathbb{R} - \{0\}$. We have $(\underline{b}_1(a), \underline{b}_2(a)) = (b_a, 0)$. Choose δ sufficiently small so that for $x \in (a, a+\delta)$ we have $\underline{b}_1(x) \in (b_*, k^*)$ and $\underline{b}_1^2(x)x - \underline{b}_1(x)\sigma^2(1-c) \in (\underline{l}_1, \underline{l}_2)$.

We argue that $f(x) = \frac{1}{\sigma^2 c\pi}\underline{b}_2(x) > 0$ for all $x \in (a, a+\delta)$. Suppose $x_0 \in (a, a+\delta)$ is such that $\underline{b}_2(x_0) = 0$. Letting $\hat{b} = \underline{b}_1(x_0)$, we have $x(\hat{b}) = x_0$. It is obvious that $\hat{b} \neq b_a$, and if $\hat{b} \in (b_*, b_a)$, then $x_0 < a$, a contradiction. Therefore $\hat{b} \in (b_a, k^*)$, and hence, $x'(\hat{b}) < 0$. For any $b \in (b_a, k^*)$ we have from (1.1′)

$$b = 1 + \sigma^2 c b m_H(b^2 x(b) - b\sigma^2(1-c)),$$

and therefore differentiating implicitly with respect to b we get

$$x'(b) = \frac{1 - \sigma^2 c b^2 m'_H(b^2 x(b) - b\sigma^2(1-c))(2bx(b) - \sigma^2(1-c))}{\sigma^2 c b^4 m'_H(b^2 x(b) - b\sigma^2(1-c))} < 0. \tag{4.1}$$

Since b is real, we have

$$\sigma^2 c b^2 m'_H(b^2 x(b) - b\sigma^2(1-c)) = \sigma^2 c b^2 \int \frac{dH(t)}{(t - (b^2 x(b) - b\sigma^2(1-c))^2}$$

$$= \sigma^2 c \int \frac{dH(t)}{\left(\frac{t}{b} - bx(b) + \sigma^2(1-c)\right)^2} = G(b),$$

and therefore

$$x'(b) = \frac{1 - G(b)(2bx(b) - \sigma^2(1-c))}{b^2 G(b)} < 0. \tag{4.2}$$

Let $z = z_1 + iz_2 \in \mathbb{C}^+$ and $b(z) = b_1(z) + ib_2(z) \equiv b_F(z)$. From (2.7) we get

$$\frac{z_2}{b_2(z)} = \frac{1 - G(b(z))(2b_1(z)z_1 - \sigma^2(1-c)) + b_2(z)z_2 G(b(z))}{b_1^2(z) G(b(z))} > 0. \tag{4.3}$$

Letting $z \to x_0$ we have $b(z) \to \hat{b}$ and therefore (4.3) gives

$$\frac{1 - G(\hat{b})(2\hat{b}x_0 - \sigma^2(1-c))}{\hat{b}^2 G(\hat{b})} \geq 0,$$

a contradiction of (4.2). Therefore, $\underline{b}_2(x_0) > 0$, and hence $f(x) > 0$ for all $x \in (a, a+\delta)$.

THEOREM 4.2. $x''(b_a) < 0$.

Proof. Since $x'(b_a) = 0$, we have, by Theorem 4.1, that b_a is a relative maximum of x. Therefore, $x''(b_a) \leq 0$. Since the first non-vanishing derivative of a function at a relative extreme must be of even order, we will assume $x''(b_a) = 0$ and $x'''(b_a) = 0$, and proceed to show a contradiction.

Let $w \equiv b^2 x(b) - b\sigma^2(1-c)$, $w_a \equiv b_a^2 a - b_a\sigma^2(1-c)$, $d \equiv 2bx(b) - \sigma^2(1-c)$, $d_a \equiv 2b_a a - \sigma^2(1-c)$, and define

$$A_j = \int \frac{dH(t)}{(t - w_a)^j} \qquad \text{for} \quad j = 2, 3, 4$$

so that $m'_H(w_a) = A_2$, $m''_H(w_a) = 2A_3$, and $m'''_H(w_a) = 6A_4$. Writing (1.1′) as

$$\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right) = m_H(w), \tag{4.4}$$

and differentiating implicitly with respect to b three times results in the following three equations

$$\frac{1}{\sigma^2 c b^2} = m'_H(w)(d + b^2 x'(b))$$

$$\frac{-2}{\sigma^2 c b^3} = m''_H(w)(d + b^2 x'(b))^2 + m'_H(w)(2x(b) + 4bx'(b) + b^2 x''(b))$$

$$\frac{6}{\sigma^2 c b^4} = m'''_H(w)(d+b^2x'(b))^3 + 3m''_H(w)(d+b^2x'(b))(2x(b)+4bx'(b)+b^2x''(b)) + m'_H(w)(6x'(b) + 6bx''(b) + b^2 x'''(b)).$$

Now, we evaluate these equations at the point b_a and use the assumption that the first three derivatives of x are zero to get the following three equations in terms of the A_j's

$$d_a A_2 = \frac{1}{\sigma^2 c b_a^2}$$

$$d_a^2 A_3 + aA_2 = -\frac{1}{\sigma^2 c b_a^3}$$

$$d_a^3 A_4 + 2ad_a A_3 = \frac{1}{\sigma^2 c b_a^4}.$$

Note that the first equation implies $d_a > 0$. Solving for A_3 and A_4 we get

$$A_3 = -\frac{1}{\sigma^2 c}\left(\frac{a}{d_a^3 b_a^2} + \frac{1}{d_a^2 b_a^3}\right)$$

and

$$A_4 = \frac{1}{\sigma^2 c}\left(\frac{2a^2}{d_a^5 b_a^2} + \frac{2a}{d_a^4 b_a^3} + \frac{1}{d_a^3 b_a^4}\right).$$

Writing $w_a = d_a b_a - b_a^2 a$ and $A_3 = \int \frac{t}{(t-w_a)^4} dH(t) - w_a A_4$ we get

$$0 < \int \frac{t}{(t-w_a)^4} dH(t) = w_a A_4 + A_3 = \frac{-2a^3 b_a^4}{\sigma^2 c d_a^5 b_a^4},$$

a contradiction since a and d_a are both positive. Therefore, $x''(b_a) < 0$.

We now show that the density f resembles a square root function in a neighborhood to the right of a.

Since $m_H'(w_a) \neq 0$, there exists a neighborhood $W \subset \mathbb{C}$ of w_a on which m_H is one-to-one, and hence has an analytic inverse. Let $B \subset \mathbb{C}$ and $U \subset \mathbb{C}$ be neighborhoods of b_a and a, respectively. Define

$$W_0 \equiv \{w \in \mathbb{C} : w = b^2 z - b\sigma^2(1-c) \text{ for } b \in B \text{ and } z \in U\}.$$

Choose B and U sufficiently small so that $B \cap \mathbb{R} \subset (k_1, k_2)$, $W_0 \subset W$ and $W_0 \cap \mathbb{R} \subset (l_1, l_2)$. Then $m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right)$ is an analytic mapping from B to W_0. For $b \in B$ define

$$z(b) = \frac{1}{b^2} m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right) + \frac{1}{b}\sigma^2(1-c). \tag{4.5}$$

Therefore, if $b \in B \cap \mathbb{R}$, we have $z(b) = x(b)$, and hence $z'(b_a) = x'(b_a) = 0$ and $z''(b_a) = x''(b_a) < 0$. By Theorem 10.32 of Rudin [6], there is a neighborhood $V \subset B$ of b_a and a function ϕ, analytic in V, such that

$$z(b) - a = (\phi(b))^2 \qquad \text{for all} \quad b \in V, \tag{4.6}$$

ϕ' has no zero in V, and ϕ is an invertible mapping of V onto a disc centered at the origin. We then have $\phi(b_a) = 0$, $\phi'(b_a) \neq 0$, and computing the first two derivatives on both sides of (4.6) we get

$$z'(b) = 2\phi(b)\phi'(b)$$

and

$$z''(b) = 2[\phi'(b)]^2 + 2\phi(b)\phi'(b).$$

Therefore

$$0 > z''(b_a) = 2[\phi'(b_a)]^2, \tag{4.7}$$

and hence $\phi'(b_a)$ must be purely imaginary. Write $\frac{1}{\phi'(b_a)} = i\alpha$, where $\alpha \in \mathbb{R}$ is nonzero.

Let $\delta > 0$ be small enough so that f is positive over $(a, a+\delta)$ and $(a, a+\delta) \subset U \cap \mathbb{R}$. Fix $x \in (a, a+\delta)$. Since $\underline{m}(x)$ satisfies (1.1) for $z = x$ we immediately have

$$x - a = [\phi(\underline{b}(x))]^2.$$

Since $x > a$, we may take the square root of both sides to get

$$\phi(\underline{b}(x)) = \sqrt{x-a},$$

where we assume that $\phi(\underline{b}(x))$ is the positive root. Let Γ be the inverse of ϕ on V. Then $\Gamma(0) = b_a$ and $\Gamma'(0) = \frac{1}{\phi'(b_a)} = i\alpha$, and expanding Γ about 0 we have

$$\underline{b}(x) = \Gamma(\sqrt{x-a}) = \Gamma(0) + \Gamma'(0)\sqrt{x-a} + \text{(higher order terms)}$$

$$= b_a + i\alpha\sqrt{x-a} + \text{(higher order terms)}.$$

Therefore

$$\underline{b}_2(x) = \sqrt{x-a}(\alpha + \text{(higher order terms)}),$$

and hence, for $x \in (a, a+\delta)$, we have expressed $f(x) = \frac{1}{\sigma^2 c\pi}\underline{b}_2(x)$ as an analytic function of $\sqrt{x-a}$. This is a stronger result that what was proven for the density in Silverstein and Choi [9], although the same method used here may be applied to that case and yield the same strong result.

The a we used was a left end-point of S_F. If a were a right end-point of S_F, the analysis would differ only slightly from what we have done here. In that case we assume that b_a is a relative minimum of $x(b)$, and therefore (4.7) becomes

$$0 < z''(b_a) = 2[\phi'(b_a)]^2,$$

giving that $\phi'(b_a)$ is nonzero and real. Write $\frac{1}{\phi'(b_a)} = \alpha$. Let $\delta > 0$ be small enough so that f is positive over $(a-\delta, a)$ and $(a-\delta, a) \subset U \cap \mathbb{R}$. Fixing $x \in (a-\delta, a)$, we again have

$$x - a = [\phi(\underline{b}(x))]^2,$$

and hence, since $x < a$, we get

$$\phi(\underline{b}(x)) = i\sqrt{|x-a|},$$

where the square root is assumed to be positive. Again letting Γ be the inverse of ϕ on V we have $\Gamma(0) = b_a$ and $\Gamma'(0) = \alpha$. Expanding Γ about 0 we get

$$\underline{b}(x) = \Gamma(i\sqrt{|x-a|}) = \Gamma(0) + i\Gamma'(0)\sqrt{|x-a|} + \text{(higher order terms)}$$

$$= b_a + i\alpha\sqrt{|x-a|} + \text{(higher order terms)},$$

and therefore

$$\underline{b}_2(x) = \sqrt{|x-a|}(\alpha + \text{(higher order terms)}).$$

5. An Example and Application

In this section we graphically analyze the limiting density and the procedure for finding S_F for a particular example of F. We compare the results of a simulation to our density graph, and use the comparisons to analyze the problem of signal detection in array signal processing.

As noted earlier, F is nonrandom and only depends on the distribution H and the constants c and σ. We construct our example by letting $c = .1$ and $\sigma = 1$ and taking H to be discrete with mass .2, .4, and .4 at the respective values 0, 3, and 10.

In section 3 we described a method by which S_F may be obtained. From each interval $I_H \subset S_H^c$ we construct a well-defined function x given (in terms of $b = 1 + \sigma^2 cm$) by

$$x(b) = \frac{1}{b^2} m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right) + \frac{1}{b}\sigma^2(1-c) \tag{5.1}$$

for b in some interval $(k_1, k_2) \subset (0, \infty)$ prescribed by I_H. We graph this function and remove the intervals along the vertical axis where the graph is increasing. We repeat this procedure for each interval $I_H \subset S_H^c$, and the set of points that have not been removed from the vertical axis will be S_F.

For our example, S_H^c is composed of the four intervals $I_{(i)} = (-\infty, 0)$, $I_{(ii)} = (0, 3)$, $I_{(iii)} = (3, 10)$, and $I_{(iv)} = (10, \infty)$, and therefore we have four functions given by (5.1). The graphs of these four functions (given as $x(m)$), obtained using Newton's method, are shown in Figure 1 (a). The thick lines on the vertical axis represent S_F. As noted in sections 3, we see that the intervals on the vertical axis where the graphs are increasing do not overlap each other from one function to the next.

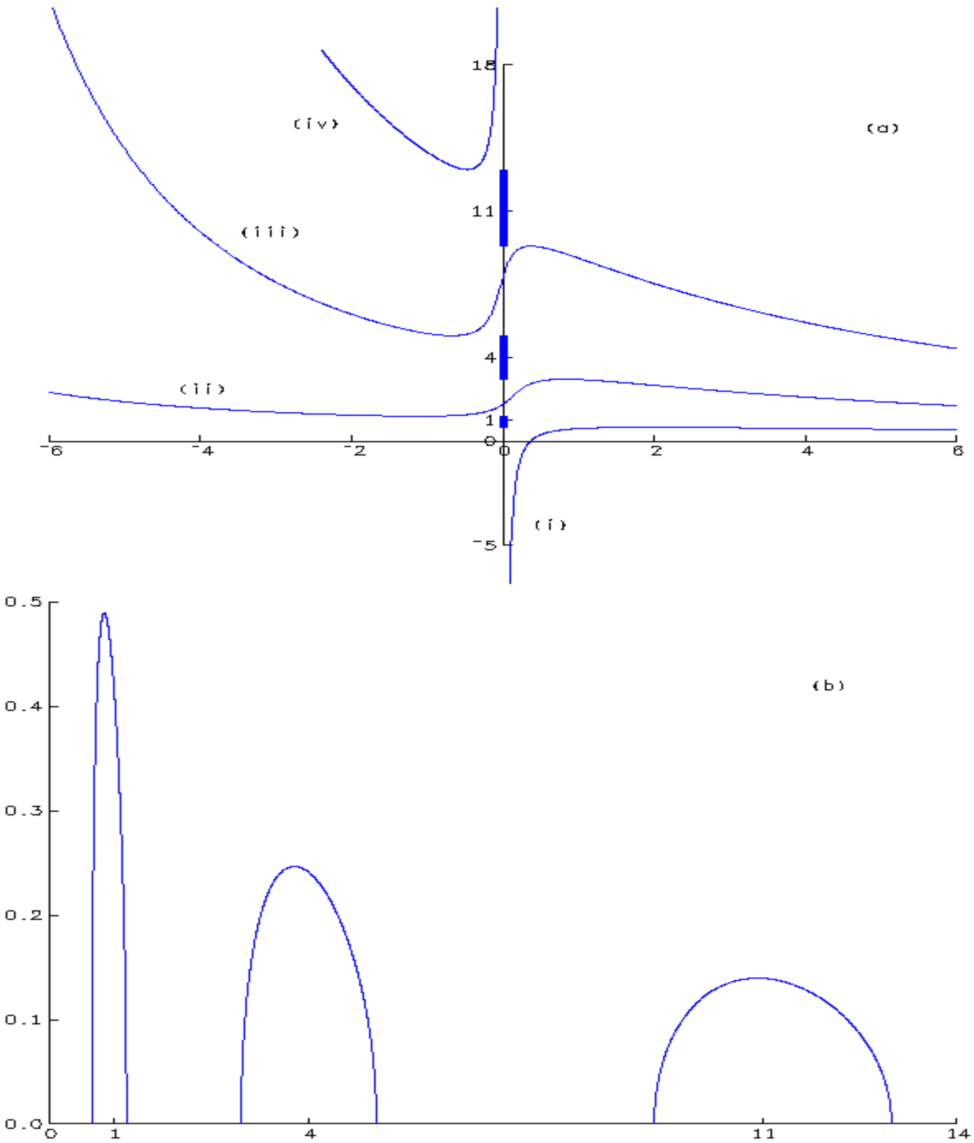

FIG1.

Once we have obtained S_F it is a simple matter of applying Newton's method to equation (1.1) with $z = x$ and $m = \underline{m}(x)$ to numerically obtain the density $f(x) = \frac{1}{\pi}\mathcal{I}m\,\underline{m}(x)$ for each $x \in S_F$. Figure 1 (b) shows the graph of the limiting density f. Note that when positive, f is a smooth function, and, at the boundary of its support, f goes down vertically to the x-axis, thus behaving in a similar fashion to a square root.

Recall that when $c \downarrow 0$, F will converge to the distribution of a random variable $Y+\sigma^2$, where Y has distribution H. For our example, as $c \downarrow 0$, F will converge to the discrete distribution having mass .2, .4, and .4 at the respective points 1, 4, and 11. It is evident that our choice of $c = .1$ is small enough to see the mass beginning to accumulate around 1, 4, and 11.

In Figure 2 we have overlaid the density graph with a histogram and a scatterplot of the eigenvalues of a simulation of the matrix C_n. We choose $n = 200$ so that, since $c = .1$, $N = 2000$. We construct R_n in a deterministic manner so that the e.d.f. of the eigenvalues of $\frac{1}{N}R_nR_n^*$ is exactly H, and we let the entries of X_n be i.i.d. standardized Gaussian. We see that the histogram of the eigenvalues of C_n follows the shape of the density and the scatterplot, with each eigenvalue marked by the symbol '$\circ$', stays close to S_F. The eigenvalues exhibit a clear separation into three distinct groups clustering near the points 1, 4, and 11. In fact, the distribution of the eigenvalues among the three groups is, from left to right, .2, .4, and .4. That is, of the 200 eigenvalues, 40 are in the first group, 80 are in the second group, and 80 are in the last group.

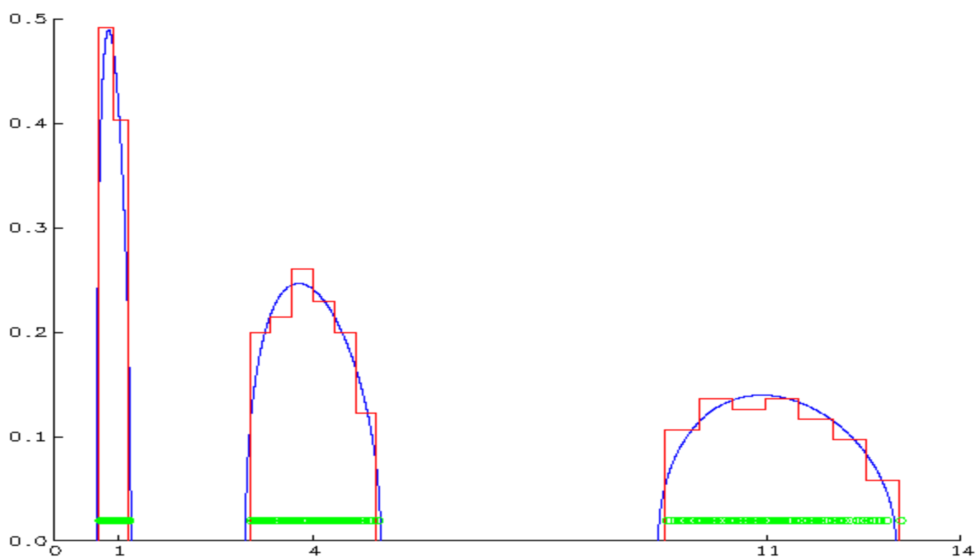

FIG2.

We use this example to illustrate the connection to the detection problem in array signal processing, where an array of n sensors receives signals transmitted by an unknown number $q < n$ of sources with unknown locations in a noise-filled environment. The goal is mainly to identify the number of sources (signal detection) and their direction of arrival (DOA). The model is given by an $n \times N$ matrix $Y_n = R_n + \sigma X_n$ in which the columns represent N "snapshots" (samples) of the received signals. The matrix R_n represents the pure signal information and contains values detailing sensor orientation, the signal values at the source, and components such as steering vectors which provide information on the unknown direction of arrival of the signals. The signals are commonly assumed to be stationary ergodic processes. The matrix X_n represents additive noise (variance σ^2 unknown) that contaminates the signal during transmission and processing. The entries of X_n are assumed to be i.i.d. standardized random variables. If the population matrix $S_n + \sigma^2 I$ ($S_n \equiv E\frac{1}{N}R_nR_n^*$) were known, or at least adequately approximated, then using the MUSIC (multiple signal classification) algorithm, as presented in Schmidt [7], one could determine the number of sources and, depending on the accuracy of the approximation, their direction of arrival. The sample covariance matrix $C_n = \frac{1}{N}Y_nY_n^*$ is used to estimate the population matrix, however, as stated in the introduction, if the number of sensors, n, is large then it may not be possible to collect enough samples to adequately estimate it. In this case, limiting results on the eigenvalues of C_n can aid in the detection problem: determining the number of sources. As noted in Schmidt [7], if $q < n$ then S_n is singular with $n - q$ zero eigenvalues. Therefore the $n - q$ smallest eigenvalues of the matrix $S_n + \sigma^2 I$ are equal to σ^2. These are called the "noise" eigenvalues, and the q larger eigenvalues are called the "signal" eigenvalues. Therefore, obtaining the value of q, the number of sources, can be accomplished by determining the multiplicity of the noise eigenvalues. From this it is clear that limiting results on the eigenvalues of the sample covariance matrix C_n can play an important role in signal detection. Indeed, if it can be shown that, for large n, the eigenvalues of C_n display this "splitting" into groups of smaller and larger eigenvalues with the correct number of eigenvalues in each group corresponding to the noise and signal eigenvalues, then determining the number of sources can be accomplished with fewer samples than needed to approximate the population matrix itself. It will only require enough samples for the eigenvalues of C_n to separate into distinct, separate clusters.

Results of this type were proven for a different class of matrices in Bai and Silverstein [1], [2] with the first paper showing that, for n large, no eigenvalues appear where they should not, i.e., outside the support of the limiting distribution, and the second paper showing that, for n large, each interval of the support contains the correct number of eigenvalues. As yet, there are no such results proven for our limiting distribution, but from simulations it appears that similar results hold true for our case as well.

In the simulation above the number of sensors is 200, sample size is 2000, the (unknown) number of sources is 160, and $\sigma^2 = 1$. Since R_n was artificially constructed so that $\frac{1}{N} R_n R_n^*$ has only two distinct nonzero eigenvalues, our example is limited in its comparison to an actual signal detection problem. Even so, this example is useful for illustrative purposes. The scatterplot shows a clear separation of the 40 noise eigenvalues from the 160 signal eigenvalues. The value $c = .1$ is certainly small enough to see the separation of the support of F into disjoint intervals. In fact, by analyzing the density for different values of c, we discover that the separation of the smaller eigenvalues from the larger ones occurs when c is approximately .37555. Therefore, for a particular value of n, it would take less than $3n$ samples for separation of the support to occur. This number is substantially smaller than the number of samples required to adequately approximate the population matrix $S_n + \sigma^2 I$ using conventional multivariate inference methodology.

Further research is needed to make rigorous the mathematical arguments for exact eigenvalue separation in our information-plus-noise model.

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